Model Structures on Topological Spaces

Andrew Ronan

Contents

1 Introduction

The goal of this essay is twofold. Firstly, we would like to generalise the small object argument, for sets of maps with sequentially small domains, to arbitrary cardinals. Secondly, we would like to derive some classical model structures on the categories of simplicial sets and topological spaces. The set theory required to meet the first objective is developed in Section 3. We then describe the small object argument for arbitrary cardinals in Section 4. For our second objective, we start in Section 2 with a number of results on model categories which are either of stand-alone interest or will help us on our way later in the essay. For example, we prove the equivalence of a selection of different definitions of a Quillen equivalence, which we use later in the essay to deduce the equivalence of the Quillen model structure on simplicial sets and the Quillen model structure on topological spaces. In Section 5, we derive the Quillen model structure on simplicial sets, making good use of the result that the realisation of a minimal fibration is a Serre fibration. In fact, in Theorem 5.3.1, we prove the stronger result, of Fritsch and Piccinini, that the realisation of a minimal fibration is a Hurewicz fibration in the category of compactly generated spaces. In Section 6, we begin by proving that the Quillen model structure on topological spaces is proper. We then move on to deriving the Hurewicz model structure, proving the factorisation axiom by using a direct argument analogous to the mapping path space construction. We then end the essay with a discussion of the mixed model structure, due to M. Cole, induced by the Quillen and Hurewicz model structures.

Notations and Conventions

We will assume that all generic categories are bicomplete, unless otherwise stated.

2 Model Categories

In this section, we prove a small compilation of results concerning model categories which are either of stand-alone interest or will help us on our way later in this essay. We will generally assume the basics of model category theory as presented in either [5] or Chapters 7 and 8 of [6]. However, there is some overlap between the material presented here and the material presented in those references. Following Chapter 14 of [2], we start by introducing a different definition of a model category and then proving that this definition is indeed equivalent to the usual one - using an observation of Joyal and Tierney (Proposition 7.8, [9]). We then proceed to study the homotopy category of a model category, introducing Quillen adjunctions and equivalences. The loosening of the definition of a Quillen adjunction given in Lemma 2.2.1 is due to D.Dugger ([10], Corollary A.2). Finally, following [6], we introduce proper model categories and prove that any model category in which all objects are cofibrant is left proper.

2.1 Weak Factorisation Systems

We begin with the following definition:

Definition: If $\mathcal{L}$ is a class of maps, let $\mathcal{L}^{\boxslash}$ denote the class of maps with the RLP with respect to $\mathcal{L}$. If $\mathcal{R}$ is a class of maps, let ${}^{\boxslash}\mathcal{R}$ denote the class of maps with the LLP with respect to $\mathcal{R}$. Write $\mathcal{L}\boxslash\mathcal{R}$ if $\mathcal{L}$ has the LLP with respect to $\mathcal{R}$.

Observe that ${}^{\boxslash}\mathcal{R}$ is closed under pushouts, retracts and countable composites. We will see later that it is also closed under transfinite composites (Lemma 4.1.3).

Definition: A weak factorisation system in M is an ordered pair $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{L} = {}^{\boxslash}\mathcal{R}$, $\mathcal{R} = \mathcal{L}^{\boxslash}$, and, if f is a morphism of M, then $f = pi$ for some $p \in \mathcal{R}$ and $i \in \mathcal{L}$.

We can now state the definition of a model structure on a category M.

Definition: A model structure on M consists of classes $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ of maps in M such that:

i) $\mathcal{W}$ has the two out of three property,

ii) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system,

iii) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system.

We call an element of $\mathcal{W}$ a weak equivalence, an element of $\mathcal{C}$ a cofibration and an element of $\mathcal{F}$ a fibration.

Since classes of maps defined by lifting properties are closed under retracts, the equivalence of this definition of a model category and the usual one is mostly clear. The only complication that arises is in proving that the class of weak equivalences of a model category, as defined above, is closed under retracts:

Lemma 2.1.1: *If M is a model category, then $\mathcal{W}$ is closed under retracts.*

Proof: Suppose that f is a retract of a weak equivalence g. We can factor f as pi, where i is an acyclic cofibration and p is a fibration. Form the following diagram, where the top left square is a pushout, all rows compose to 1 and $g = p^{'}i^{'}$:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i"] & A \arrow[r, "r"] \arrow[d, "i'"] & X \arrow[d, "i"] \\ Z \arrow[r] \arrow[d, "p"] & P \arrow[r] \arrow[d, "p'"] & Z \arrow[d, "p"] \\ Y \arrow[r, "v"] & B \arrow[r, "u"] & Y \end{tikzcd}$$

Then $i^{'}$ is an acyclic cofibration, since it is a pushout of i, and so $p^{'}$ is a weak equivalence. Therefore, by considering the bottom rectangle, we have reduced to the case where f is a fibration. For this case factor $g = pi$, where p is an acyclic fibration and i is an acyclic cofibration. Now consider the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dd, "f"] \arrow[dr, dashed, "h"] & A \arrow[r, "r"] \arrow[d, "i"] & X \arrow[dd, "f"] \\ & Z \arrow[d, "p"] \arrow[ur, dashed, "k"] & \\ Y \arrow[r, "s"] & B \arrow[r, "r"] & Y \end{tikzcd}$$

where $h = is$ and k is a lift in the square with LHS the acyclic cofibration i and RHS the fibration f. We have $kh = 1$ and so f is a retract of the acyclic fibration p and, hence, an acyclic fibration itself. $\square$

2.2 The Homotopy Category of a Model Category

Recall that the localisation $\gamma : M \to \mathrm{Ho}(M)$ of a model category M with respect to the class of weak equivalences can be constructed homotopically and satisfies the universal property that any functor from M which takes weak equivalences to isomorphisms factors uniquely through the homotopy category. If $F : M \to D$ is any functor, then the left derived functor of F, if it exists, is the best approximation to a factorisation of F through the homotopy category "from the left". If F is a functor that takes acyclic cofibrations between cofibrant objects to isomorphisms, then the left derived functor can be shown to exist. In effect, it is the factorisation of the functor obtained by applying cofibrant approximation before F through the homotopy category. We also have the notion of a total left derived functor of a functor between model categories. We have the following definition:

Definition: Let M and N be model categories. A Quillen adjunction is an adjunction $F \dashv G$, where $F : M \to N$, such that F preserves cofibrations and acyclic cofibrations.

It can be shown that, if (F, G) is a Quillen pair, then both $\mathbb{L}F$ and $\mathbb{R}G$ exist and form an adjoint pair between the homotopy categories of M and N. There are a number of equivalent requirements for an adjoint pair to be a Quillen adjunction - for example, it is equivalent to require that G preserves fibrations and acyclic fibrations. In fact, we have:

Lemma 2.2.1: *If $F \dashv G$ is an adjunction between model categories, then, if F preserves acyclic cofibrations and cofibrations between cofibrant objects, $F \dashv G$ is a Quillen adjunction.*

Proof: Since F preserves acyclic cofibrations, G preserves fibrations and so it is enough to show that, if $f : X \to Y$ is an acyclic fibration in N, then Gf is a weak equivalence. We will use the result that a map between fibrant objects P and Q is a weak equivalence iff for every cofibrant A it induces an isomorphism $\pi(A, P) \cong \pi(A, Q)$.

Since Gf is a fibration, both GX and GY, with the obvious choice of maps, are fibrant in the over-category $(M \downarrow GY)$. If $g : A \to GY$ is a map in $(M \downarrow GY)$, with A cofibrant, then we have a diagram:

$$\begin{tikzcd} \star \arrow[r] \arrow[d] & GX \arrow[d, "Gf"] \\ A \arrow[r, "g"'] \arrow[ur, dashed] & GY \end{tikzcd}$$

where, viewed as a square in M, the lift exists since the lift exists in the adjoint square, because F preserves cofibrations between cofibrant objects. It is also clear that this implies the lift exists in the overcategory, which implies surjectivity of the map $(Gf)_* : \pi(A, GX) \to \pi(A, GY)$. For injectivity, suppose that $g, h : A \to GX$ are maps in $(M \downarrow GY)$ such that $Gf \circ g$ is homotopic to $Gf \circ h$. Then, we can choose a very good cylinder object and construct a diagram:

$$\begin{tikzcd} A \coprod A \arrow[r, "g \coprod h"] \arrow[d, "i"'] & GX \arrow[d, "Gf"] \\ E \arrow[r, "H"'] \arrow[ur, dashed, "K"'] & GY \end{tikzcd}$$

where the lift exists using the same argument as with the previous square, noting that, since A is cofibrant, i is a cofibration between cofibrant objects. This proves injectivity and, therefore, Gf is a weak equivalence in $(M \downarrow GY)$ and, hence, in M. $\square$

Given a Quillen adjunction, a natural question to ask is when is the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ an equivalence of homotopy categories? We have the following definition and criterion:

Definition: A Quillen adjunction $F \dashv G$ is a Quillen equivalence if $\mathbb{L}F \dashv \mathbb{R}G$ is an adjoint equivalence of categories.

Lemma 2.2.2: *Let $F \dashv G$ be a Quillen adjunction. Then the following are equivalent:*

i) $F \dashv G$ is a Quillen equivalence,

ii) a map $f : X \to GY$ in M, where X is cofibrant and Y is fibrant, is a weak equivalence iff its adjoint $\tilde{f} : FX \to Y$ is a weak equivalence in N,

iii) F reflects weak equivalences between cofibrant objects and the composite $\epsilon_Y \circ Fp_{GY} : FQGY \to Y$ is a weak equivalence for all fibrant Y.

If, moreover, F creates the weak equivalences in M, the following statement can be added:

iv) $\epsilon_Y : FGY \to Y$ is a weak equivalence for all fibrant Y.

Proof: i) $\implies$ ii) Recall that, if X is a cofibrant object of M, then the identity map $\mathbb{L}FX \to \mathbb{L}FX$ corresponds to a homotopy class of maps $FX \to RFX$ which corresponds, under the adjunction $F \dashv G$, to a homotopy class of maps $X \to GRFX$ which, in turn, corresponds to a map $X \to \mathbb{R}G\mathbb{L}FX$. Explicitly, it corresponds to the homotopy class of $\tilde{i}_{FX}$ in $\pi(X, GRFX)$. Therefore, the unit, of the derived adjunction, is an isomorphism for all X iff $\tilde{i}_{FX}$ is a weak equivalence for all cofibrant X. Dually, the counit is an isomorphism iff $\tilde{p}_{GY}$ is a weak equivalence for all fibrant Y.

Now let $f : X \to GY$ be a weak equivalence in M, with X cofibrant and Y fibrant. Then, let $h : X \to QGY$ be a cofibrant approximation of f, so $f = p_{GY}h$. We have $\tilde{f} = \tilde{p}_{GY} \circ Fh$, which is a weak equivalence since we've shown $\tilde{p}_{GY}$ is one already, and F preserves weak equivalences between cofibrant objects. Dually, if $f : FX \to Y$ is a weak equivalence, with X cofibrant and Y fibrant, then so is its adjoint.

ii) $\implies$ iii) Firstly, $\epsilon_Y \circ Fp_{GY} = \tilde{p}_{GY}$ and, so, since p_{GY} is a weak equivalence from a cofibrant object to GY, where Y is fibrant, $p_{\tilde{GY}}$ is a weak equivalence. If $f : X \to Y$ is a map between cofibrant objects such that Ff is a weak equivalence, then $i_{FY} \circ Ff$ is also a weak equivalence. Hence, the adjoint is a weak equivalence, but the adjoint is $\tilde{i}_{FY} \circ f$ and $\tilde{i}_{FY}$ is a weak equivalence since i_{FY} is. Therefore, f is a weak equivalence.

iii) $\implies$ i) We've seen in the proof of i) $\implies$ ii) that the counit of the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ at a fibrant object Y, corresponds to $\tilde{p}_{GY}$, so if this is a weak equivalence for all fibrant Y, we must have that the counit is an isomorphism. We claim that $\mathbb{L}F$ reflects isomorphisms. Indeed, it is enough to show that it reflects isomorphisms between bifibrant objects and, in that case, maps $A \to B$ in $\mathrm{Ho}(M)$ correspond to homotopy classes of maps $A \to B$ in M. If $\mathbb{L}F(f) : FA \to FB$ is an isomorphism, then, if g is a representative of the homotopy class defining f, Fg is a weak equivalence and so g is a weak equivalence which implies that f is an isomorphism as required. Finally, for any X, we have $\epsilon_{QX} \circ \mathbb{L}F\eta_X = 1$ and, so, the unit is an isomorphism for all X as well.

In the case where F creates the weak equivalences in M, the equivalence of iv) and iii) is immediate. $\square$

2.3 Proper Model Categories

Finally, for this section, we turn our attention to proper model categories. In particular, we'll prove just enough to deduce properness for many of the examples of model categories we'll come across later in this essay. We begin with the definition:

Definition: A model category M is said to be left proper if the pushout of a weak equivalence along a cofibration is a weak equivalence. It is said to be right proper if M^{op} is left proper, and proper if it is both left and right proper.

Proposition 2.3.1: *If M is a model category, then a pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence.*

Proof: Suppose that we have a pushout square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "j"] \\ Y \arrow[r, "g"'] & P \end{tikzcd}$$

where f is a weak equivalence and i is a cofibration. We'll show that for any fibrant Z, $g^* : \pi(P, Z) \to \pi(Y, Z)$ is an isomorphism.

For surjectivity, suppose that $u : Y \to Z$ is a map. Then, since f is a weak equivalence between cofibrant objects, there exists $v : X \to Z$ such that $vf \simeq ui$. Since i is a cofibration, this implies that $u \simeq \phi$ for some ϕ with $\phi i = vf$ and so the universal property of the pushout defines a map $\psi : P \to Z$ such that $\psi g = \phi$.

For injectivity, suppose that we have a diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "H"] \arrow[d, "g"] & D \arrow[d, "p"] \\ X \arrow[r, "j"'] & P \arrow[r, "u \times v"'] & Z \times Z \end{tikzcd}$$

where $p : D \to Z \times Z$ is the second map in a very good path object for Z, and so is a fibration. Therefore, in the category $(M \downarrow Z \times Z)$, D is fibrant. Since f is a weak equivalence, there exists a unique, up to homotopy, map $K : X \to D$ such that $Kf \simeq Hi$ in the over-category. Since, i is a cofibration, there is a homotopy $H \simeq L$ in $(M \downarrow Z \times Z)$ such that $Li = Kf$, and observe that the right hand square still commutes with H replaced by L. Therefore, L and I define a map $G : P \to D$ by the universal property of the pushout in M. We have that $pG = u \times v$ due to K being defined in the over-category and the universal property of the pushout. Therefore, $u \simeq v$ and we have injectivity. $\square$

Corollary 2.3.2: *If M is a model category in which every object is cofibrant, then M is left proper.* $\square$

3 Set Theory and Ordinals

In this section, we will give a brief introduction to ordinals and cardinals that will cover the results we will need later on in our discussion of the small object argument. We will follow closely Chapter 2 of [4]. We start the section off with a proof of the equivalence of the axiom of choice and Zorn's Lemma, on the way giving a quick definition of the class of ordinal numbers. We then turn to the study of well-orderings in general, culminating in a proof that every well-ordered set is isomorphic to a unique element of the class of ordinal numbers. Finally, we introduce cardinals and prove a few of their basic properties.

We shall not delve too far into the axiomatics of set theory, but to emphasise their importance we describe below how to use the axioms to prove that a set cannot be a member of itself, a result that will play a central role in our discussion of ordinals.

Axiom of Foundation: If $A \neq \emptyset$, then there exists $u \in A$ such that $u \cap A = \emptyset$.

Lemma: If A is a set, then $A \notin A$.

Proof: Consider the set $\{A\}$ containing just one element, A. Then the axiom of foundation implies that $A \cap \{A\} = \emptyset \implies A \notin A$. $\square$

3.1 The Axiom of Choice, Zorn's Lemma and well-orderings

We start with the following basic definitions:

Definition: A relation $\sim$ on a set S is called a:

i) preorder if it is transitive and reflexive, ie for any $a \in S$, $a \sim a$,

ii) a partial order if it is a preorder and $a \sim b$ and $b \sim a \implies a = b$,

iii) a total order if it is a partial order where any two elements of S are related,

iv) a well order if it is a total order and every subset T of S has a minimal element, ie an element $a \in T$ such that for all $t \in T, a \leq t$.

Defintion: Let $(A, <)$ be a pre-ordered set. Then:

i) $m \in A$ is called a maximal element of A if whenever $a \in A$ is related to m, $a < m$,

ii) $a_0 \in A$ is called an upper bound for a subset $B \subset A$ if for all $b \in B$, $a_0 > b$,

iii) $B \subset A$ is called a chain if any two elements of B are related.

Definition: Let $(W, <)$ be a well-ordered set. Then a subset $A \subset W$ is called an:

i) ideal if $x < a$ and $a \in A \implies x \in A$,

ii) an initial interval if it is of the form $W(a) := \{b \in W | b < a, b \neq a\}$ for some $a \in W$.

It is trivial to verify that initial intervals in W are in 1-1 correspondence with elements of W and that the only ideal which is not an initial interval is W itself.

With these basic definitions out of the way, we now turn our attention to proving the equivalence of the axiom of choice and Zorn's Lemma. They key to the proof will be the following definition and lemma.

Definition: Let X be a set, let $\mathfrak{F} \subset \mathcal{P}(X)$, and let $\psi : \mathfrak{F} \to X$ be a fixed map. Then $\mathfrak{F}$ is called a ψ-tower if:

i) $\emptyset \in \mathfrak{F}$,

ii) if $\{A_\alpha | \alpha \in A\}$ is any totally ordered (by inclusion) family of sets in $\mathfrak{F}$, then $\cup_{\alpha \in A} A_\alpha \in \mathfrak{F}$,

iii) if $A \in \mathfrak{F}$, then $A \cup \{\psi(A)\} \in \mathfrak{F}$.

Lemma 3.1.1: *If $\mathfrak{F}$ is a ψ-tower, then there exists an $A \in \mathfrak{F}$ such that $\psi(A) \in A$.*

Proof: Call a set $A \in \mathfrak{F}$ ψ-ordered if it has a well-ordering such that for any $a \in A$:

i) $\{b \in A | b < a\} \in \mathfrak{F}$,

ii) $\psi(\{b | b < a\}) = a$.

So, for example, the empty set is ψ-ordered and any non-empty ψ-ordered set has minimal element $\psi(\emptyset)$. Any ψ-ordered set has a unique well-ordering satisfying i) and ii) since, if $<_1$ and $<_2$ are different such orderings, then we can consider the $<_1$-minimal element $a \in A$ such that $\{b | b <_1 a\} \neq \{b | b <_2 a\}$. We have:

$$\{b <_1 a\} = \cup_{c <_1 a} \{b \leq_1 c\} = \cup_{c <_1 a} \{b \leq_2 c\} = \{b <_2 \tilde{a}\}$$

where $\tilde{a}$ is the $<_2$-minimal element greater than each $c <_1 a$. Applying ψ shows us that $a = \tilde{a}$, a contradiction.

Now suppose that A and B are both ψ-ordered sets and that $b \in B \setminus A$. Consider the $<_A$-minimal $a \in A$ such that either $a \notin B$ or $\{c <_A a\} \neq \{c <_B a\}$. Then

$$\{c <_A a\} = \cup_{x <_A a} \{c \leq_A x\} = \cup_{x <_A a} \{c \leq_B x\} = \{c <_B \tilde{a}\}$$

where $\tilde{a}$ is the minimal element of B greater than each $x <_A a$, which is well-defined since b is such an element. Applying ψ tells us that $a = \tilde{a}$, a contradiction. Therefore, A is an initial interval of B.

It follows that the set of ψ-ordered elements of $\mathfrak{F}$ is totally ordered by inclusion and so, if A is the union of all ψ-ordered sets, then $A \in \mathfrak{F}$. Moreover, A is itself ψ-ordered with the induced well-ordering from the union (note that it is important that the inclusion of a ψ-ordered set into another is the inclusion of an initial interval, and not just order-preserving). Therefore, A is the maximal element of the set of ψ-ordered elements of $\mathfrak{F}$ and this implies that $\psi(A) \in A$ since, otherwise, $A \cup \{\psi(A)\}$ is itself a ψ-ordered set. $\square$

Definition: Much of the proof of the previous lemma continues to work even when X is a class. In particular, if $X = \underline{\text{Set}}$ is the class of all sets, $\mathfrak{F}$ is the power class of X, ie $a \in \mathfrak{F} \iff (b \in a \implies b \in X)$, and $\psi(A) = A$, then we call the class of ψ-ordered sets the **class of ordinal numbers**. The elements of this class are called ordinal numbers and should be viewed as well-ordered sets with the ordering induced by ψ.

Intuitively, ordinal numbers are sets of the form:

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, ...\}$$

Lemma 3.1.2: *The class of ordinal numbers, $\mathfrak{O}$, has the following properties:*

i) $\mathfrak{O}$ is not a set,

ii) if $a, b \in \mathfrak{O}$ then either a is an ideal of b or b is an ideal of a,

iii) if $a \in \mathfrak{O}$ and $b \in a$, then $b \in \mathfrak{O}$,

iv) if $a, b \in \mathfrak{O}$, then $a \subset b$ iff $a = b$ or $a \in b$,

v) if $a, b, c \in \mathfrak{O}$ and $a, b \in c$, then $a \leq b$ in c iff $a \subset b$,

vi) $\mathfrak{O}$ is well-ordered,

vii) if $a \in \mathfrak{O}$, then a is the set of ordinal numbers less than a

viii) if $\{a_i | i \in S\}$ is any set of ordinal numbers, then there is an ordinal number strictly greater than all of them.

Proof: i) if $\mathfrak{O}$ were a set, then, as in the proof of the previous lemma, take the union, A, of the elements of $\mathfrak{O}$, then $A \in \mathfrak{O}$ and is a maximum element so $\psi(A) = A \in A$, a contradiction,

ii) follows from the proof of the previous lemma,

iii) since a is ψ-ordered, any initial interval of a is ψ-ordered, in particular $b = \{c \in a | c < b\}$ is ψ-ordered and so an ordinal number,

iv) if a is a strict subset of b, then note that $a \cup \{a\}$ is a ψ-ordered set, so since $\mathfrak{O}$ is totally ordered by inclusion, we must have $a \in b$. On the other hand, if $a \in b$, then b is not a strict subset of a since that would imply $a \in b \in a$ which contradicts the axiom of foundation applied to the set $\{a, b\}$. Hence, $a \subset b$ since $\mathfrak{O}$ is totally ordered,

v) Since each element of c is the set of preceding elements of c, $a \leq b$ in $c \iff a = b$ or $a \in b \iff a \subset b$,

vi) if $S \subset \mathfrak{O}$ is a non-empty subset, let $s \in S$ and observe that if $a \in S$ and $a < s$, then $a \in s$ and so $\{a \in S | a < s\}$ forms a subset of s which (assuming it is non-empty) has a minimal element t since s is well-ordered and so part v) implies that t is a minimal element for S,

vii) follows from iii) and iv)

viii) The union of ordinal numbers is itself ψ-ordered and so an ordinal number and so if A is the union of the elements of the set in question, $A \cup \{A\}$ is an ordinal number strictly larger than any a_i. $\square$

Theorem 3.1.3: *The following are equivalent:*

i) The axiom of choice: if $\{A_i | i \in S\}$ is a family of sets, then there exists a function $f : S \to \cup_{i \in S} A_i$ such that for every $j \in S, f(j) \in A_j$.

ii) Zorn's Lemma: Let X be a preordered set. If each chain in X has an upper bound, then X has at least one maximal element.

iii) Every set can be well-ordered.

Proof: i) $\implies$ ii). Suppose that X does not have a maximal element. Then, if C is a chain in X, we can always find an upper bound for C which is not an element of C, since if u is an upper bound for C, there exists $v > u$ such that u is not greater than v, and so $v \notin C$. Therefore, let $\mathfrak{F}$ be the family of chains in X and ψ be a function taking each chain C to one of its upper bounds in $X \setminus C$. The construction of ψ is made possible by the Axiom of Choice. Then it is easy to see that $\mathfrak{F}$ is a ψ-tower and so the previous lemma applies to show that there is a chain C such that $\psi(C) \in C$, a contradiction.

ii) $\implies$ iii). Let S be a set, and consider pairs $(A, <_A)$ where $A \subset S$ and $<_A$ is a well-ordering on A. Let X be the set of such pairs, which is non-empty since $\emptyset$ is well-ordered. Then say that $(A, <_A) < (B, <_B)$ if $A \subset B$, the inclusion is order-preserving and, if $a \in A$ and $b <_B a$, then $b \in A$. Then $<$ is a partial order on S. If C is a chain in S, then let $(U, <)$ be the union of all elements of C with the induced ordering. Suppose that $A \subset U$ is non-empty. If $a \in A$, then $a \in P$ for some $P \in C$. If $b < a$ in A, then $b \in Q$ for some $Q \in C$ and either $P \subset Q$ or $Q \subset P$. If $P < Q$, then $b \in P$ by the definition of the ordering on S. Therefore, $b \in P$. So $\{b \in A | b < a\} \subset P$ and so has a minimal element since P is well-ordered. Then this element is also a minimal element for A, so U is well-ordered. So Zorn's Lemma applies to tell us that X has a maximal element, which must be $(S, <_S)$ for some well-ordering $<_S$ on S, since otherwise we can create a larger pair in X by adjoining a last element.

iii) $\implies$ i). If $\{A_i | i \in S\}$ is a family of sets, then there exists a well-ordering on $\cup_i A_i$ and so a function $f : S \to \cup_i A_i$ sending j to the minimal element of A_j, which is itself an element of A_j. $\square$

3.2 Well-ordered sets

We've shown that every set can be well-ordered so we will now turn our attention to the properties of well-ordered sets. We will assume that all maps between well-ordered sets are order-preserving, unless stated otherwise. We start with:

Lemma 3.2.1: *Let W be a well-ordered set, and $\Sigma \subset I(W)$ be any family with the properties:*

i) Σ is closed under unions,

ii) if $W(a) \in \Sigma$, then $W(a) \cup \{a\} \in \Sigma$,

iii) $\emptyset \in \Sigma$.

Then $\Sigma = I(W)$.

Proof: If I is a minimal ideal not in Σ, then we have $I = \cup_{a \in I} W(a) \cup \{a\}$, a contradiction. □

Lemma 3.2.2: *Let X be a well-ordered set and suppose that $i : W \to X$ is the inclusion of an ideal. Then, if $f : W \to X$ is any monomorphism, we have $f(w) > i(w)$ for all $w \in W$. In particular, there is at most one isomorphism between two well-ordered sets W and X.*

Proof: Otherwise, let w be the minimal element of W such that $f(w)$ is strictly less than $i(w)$. We have that $i(w)$ is the minimal element of X strictly greater than $i(v)$ for every $v < w$. Moreover, $f(w)$ is strictly greater than $f(v)$ for every $v < w$. So, since $f(v) > i(v)$ for every such v, we have $f(w) > i(w)$, a contradiction.
If f and g are two isomorphisms between W and X, then the same argument shows us that both $f(w) > g(w)$ and $g(w) > f(w)$ are true for all $w \in W$, and so $f = g$. □

Theorem 3.2.3: *Let W and X be well-ordered sets. Then precisely one of the following is true:*

i) there is a unique isomorphism from W to X,

ii) there is an isomorphism of W onto an initial interval of X,

iii) there is an isomorphism of X onto an initial interval of W.

Proof: The proof of the previous lemma implies that no two ideals of X are isomorphic and, therefore, no two ordinal numbers are isomorphic. Let S be the set of ideals of W isomorphic to an ordinal number. We have $\emptyset \in S$ and if $\{I_\alpha | \alpha \in S\}$ is a set of ideals in S, with I_α isomorphic to the ordinal number b_α, then the union of the b_α is an ordinal number, call it b. The uniqueness of the isomorphisms $I_\alpha \to b_\alpha$ implies that we have an isomorphism of I onto b, where $I = \cup_\alpha I_\alpha$. Moreover, if $W(a)$ is isomorphic to the ordinal number b, then $W(a) \cup \{a\}$ is isomorphic to $b \cup \{b\}$, so the first lemma of this section $\implies$ W is isomorphic to a unique ordinal number. The theorem now follows from properties of ordinal numbers. □

Corollary 3.2.4: *Any well-ordered set W is isomorphic to a unique ordinal number.*

Proof: Follows from the proof above. □

Corollary 3.2.5: *Any subset of a well-ordered set W is isomorphic to an ideal of W.*

Proof: Otherwise, W is isomorphic to an initial interval of A and so there is a monomorphism $f : W \to W$ which is not surjective, a contradiction by comparison with $1 : W \to W$, since if a is the minimal element of A not in the interval corresponding to W, then $f(a)$ is strictly less than a. $\square$

3.3 Cardinals

The ordinal numbers measure the length of an order on a set, but we are also interested in measuring the size of a set independently of any ordering of the set. For example, if ω is the ordinal number corresponding to the natural numbers and $\omega + 1$ is the ordinal formed by adjoining a last element, then, if A is a well-ordered set of ordinality ω and B is a well-ordered set of ordinality $\omega + 1$, then there is a bijection between A and B even though they have different ordinalities.

Definition: Two sets X and Y are said to have the same cardinality if there is a bijection between them. We write card $X =$ card Y if X and Y have the same cardinality. If there is an injection from X to Y, we write card $X \leq$ card Y.

It follows from the axiom of choice that if there is a surjection $f : Y \to X$, then card $X \leq$ card Y.

Definition: If X is a set, the cardinal number of X, denoted by $\aleph(X)$, is the minimal ordinal number that has the same cardinality as X.

Lemma 3.3.1: *card $X \leq$ card Y iff $\aleph(X) \leq \aleph(Y)$.*

Proof: $\aleph(X) \leq \aleph(Y) \implies \aleph(X) \subset \aleph(Y)$ and so there is an injection from X to Y as the composite of this inclusion with two bijections. On the other hand, if there is an injection from X to Y, then X is in bijection with a subset of $\aleph(Y)$ and so is in bijection with an ordinal number $\leq \aleph(Y)$, by Corollary 3.2.5. $\square$

Corollary 3.3.2: *(Bernstein-Schröder) If there is an injection from X to Y and an injection from Y to X, then there is a bijection from X to Y.*

Proof: We have $\aleph(X) = \aleph(Y)$, which gives the result. $\square$

We now establish some basic properties of the class of cardinal numbers, $\mathcal{H}$:

Lemma 3.3.3: *i) card $\mathcal{P}(X)$ is strictly greater than card X, and, therefore, every cardinal number has a cardinal number strictly greater than it,*
ii) $\aleph_0$ is the smallest infinite cardinal, and so every infinite set contains a countable subset,
iii) $\mathcal{H}$ is not a set.

Proof: i) There is an injection from X into $\mathcal{P}(X)$ and so card $X \leq$ card $\mathcal{P}(X)$. Suppose that we have a bijection $\phi : X \to \mathcal{P}(X)$. Consider $A = \{x \in X | x \notin \phi(x)\}$. Then, there exists $y \in X$ such that $\phi(y) = A$. If $y \in A$, then $y \notin \phi(y) = A$ and, if $y \notin A$, $y \in \phi(y) = A$, a contradiction. So the cardinality of $\mathcal{P}(X)$ is strictly greater than the cardinality of X.

ii) ω is the smallest ordinal number greater than any finite ordinal number and card $\omega = \aleph_0$ which is greater than any finite cardinal,

iii) If $\mathcal{H}$ were a set, we could write $\mathfrak{O}$ as the union of the sets of ordinal numbers below any given cardinal number, implying that $\mathfrak{O}$ is a set, a contradiction. $\square$

We finish this section with a discussion of cardinal arithmetic, which makes precise the notion of multiplying, summing and exponentiating cardinals. Most of the proofs in this section are straightforward and we leave them to the reader:

Definition: Suppose that S is a set and, for each $i \in S$, we have a cardinal number $\aleph_i$. Let $\{A_i | i \in S\}$ be a disjoint family of sets, where card $A_i = \aleph_i$. We define:

i) $\prod_{i \in S} \aleph_i =$ card $\prod_{i \in S} A_i$,

ii) $\sum_i \aleph_i =$ card $\cup_i A_i$,

iii) if $I = \{0, 1\}$, define $\aleph_0^{\aleph_1}$ to be the cardinality of the set of maps from A_1 into A_0.

Multiplication and addition of cardinals are both commutative and associative. The sum of $\aleph_1$ copies of the same cardinal $\aleph_0$ is equal to $\aleph_0 \aleph_1$. If $\{A_i | i \in S\}$ is a family of sets, then card $\cup_i A_i \leq$ card $\sqcup_i A_i$. The usual rules of exponentiation apply.

Lemma 3.3.4: *If* $\aleph \geq \aleph_0$*, then* $\aleph\aleph = \aleph$.

Proof: Suppose that $\aleph$ is the minimal infinite cardinal such that $\aleph\aleph > \aleph$. Since $\aleph$ is infinite, it is the union of all ordinal numbers less than $\aleph$. In fact, since $\aleph \neq \aleph_0$ it is the union of all infinite ordinals less than it and so we can assume all ordinals appearing in disjoint unions are infinite from this point forward. Hence, if A_a is a well-ordered set isomorphic to the ordinal number a, then we have:

$$\text{card}(\sqcup_{a \in \aleph} A_a) \geq \aleph$$

The set $\sqcup_a A_a$ is itself well-ordered by ordering first by the ordinal a and then by the ordering of A_a. Therefore, it has an ideal isomorphic to $\aleph$. This can either be the whole set or of the form $T = \sqcup_{a \in b} A_a \sqcup W(x)$ where $b \in \aleph$ and $W(x)$ is an initial interval of A_b. Note that $W(x)$ cannot be both non-empty and finite. Then:

$$T \times T = \sqcup_{p,q \in b} A_a \times A_b \sqcup_{a \in b} A_a \times W(x) \sqcup_{a \in b} W(x) \times A_a \sqcup W(x) \times W(x)$$

Finally, note that for any infinite cardinal M strictly less than $\aleph$, we have $MM = M$ and so $M = 2M = 3M = ... = MM$. Rearrangement via bijections of the expression on the RHS of the above equation shows that it is in bijection with T and so $\aleph\aleph = \aleph$, a contradiction. $\square$

4 The small object argument

In this section, following Chapter 10 of [6], we generalise the small object argument of [5] by allowing the domains of maps to be κ-small with respect to any cardinal κ, not necessarily $\aleph_0$.

4.1 Transfinite Composition

We begin with an easy to prove observation and a definition:

Lemma 4.1.1: *Suppose that S is a totally ordered set and T is a right cofinal subset of S ie, if $s \in S$ then $\exists t \in T$ with $t \geq s$. Then, if $F : S \to C$ is a diagram, the induced map $colim(F|_T) \to colimF$ is an isomorphism.*

Definition: A cardinal $\aleph$ is said to be regular if whenever S is a set of cardinality strictly less than $\aleph$ such for each $s \in S$ there is a set A_s with cardinality strictly less than $\aleph$, then the cardinality of $\cup_{s \in S} A_s$ is strictly less than $\aleph$.

For example, if $\aleph_0$ denotes the first infinite cardinal, which is itself regular, then, since the class of cardinals is well-ordered, we have cardinals $\aleph_1, \aleph_2, ..., \aleph_\omega$. We also have that $\aleph_\omega = \cup_{i=0}^{\infty} \aleph_i$ and this shows that $\aleph_\omega$ is not a regular cardinal. However, as a consequence of Lemma 3.3.4, it is true that any successor cardinal is regular.

Definition: If λ is an ordinal, then a λ-sequence in C is a functor $X : \lambda \to C$ such that for every limit ordinal $\gamma < \lambda$, the induced map $\text{colim}_{\beta<\gamma} X_\beta \to X_\gamma$ is an isomorphism. The composition of the λ-sequence is the map $X_0 \to \text{colim}X$.

Definition: Let D be a class of maps in C. Then a map $f : A \to Y$ is said to be a transfinite composite of maps in D if it is the composition of a λ-sequence, X, such that each map of the form $X_\beta \to X_{\beta+1}$ is in D.

Lemma 4.1.2: *If S is a set and, for each $s \in S$, $g_s : X_s \to Y_s$ is a map in C, then the coproduct $\coprod_{s \in S} g_s$ is a transfinite composite of pushouts of the g_s.*

Proof: Give a well-ordering to S and adjoin a maximal element to create $T = S \cup \{\star\}$. Then, for each $t \in T$, let $A_t = \coprod_{s<t} Y_s \coprod_{s \geq t} X_s$, $f_s : A_s \to A_{s+1}$ be the map induced by g_s and identity maps. Then we have a pushout square:

$$\begin{tikzcd} X_s \arrow[r, "g_s"] \arrow[d] & Y_s \arrow[d] \\ A_s \arrow[r, "f_s"'] & A_{s+1} \end{tikzcd}$$

and, together, the f_s define a λ-sequence whose composition is $A_0 \to A_\star$, ie the desired coproduct. $\square$

Definition: If $X_0 \to X_1 \to ... \to X_\beta \to ...$ is a λ-sequence such that each $X_\beta \to X_{\beta+1}$ is itself the composite of a λ_β-sequence, then we can combine/interpolate the sequences in a canonical way to create a μ-sequence, with $\mu \geq \lambda$, with the original λ-sequence right cofinal in the resulting μ-sequence. We call this μ-sequence the sequence obtained by interpolating the λ_β-sequences into the λ-sequence.

It follows that if a map $X \to Y$ is a transfinite composite of pushouts of coproducts of elements of D, then it is also a transfinite composite of pushouts of elements of D.

As promised earlier, we'll now prove that a transfinite composite of maps with the LLP with respect to a given map p also has the LLP with respect to p.

Lemma 4.1.3: *Let $p : Y \to Z$ be a map. Then the class of maps with the LLP with respect to p is closed under trasfinite composites.*

Proof: Suppose that $f : A \to B$ is the composition of the λ-sequence $X_0 \to X_1 \to ... \to X_\beta \to ...$ and that each $X_\beta \to X_{\beta+1}$ has the LLP with respect to p. Suppose that we have a commutative square:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "p"] \\ B \arrow[r, "h"'] & Z \end{tikzcd}$$

Consider the set S of pairs (I, ϕ) where I is an ideal of λ and $\phi : \mathrm{colim}_I X \to Y$ is such that $\phi u_I = g$ and $hv_I = p\phi$, where $f = u_I v_I$, with u_I the map from A to the colimit under I and v_I the induced map from the colimit under I to the colimit of X, which is B. Then S is partially ordered by $(I, \phi) \leq (J, \psi)$ if $I \subset J$ and $\phi = \psi|_I$. If $\{(I_t, \phi_t)|t \in T\}$ is a totally ordered subset of S, then viewing the ϕ_t as cones under ideals of λ which agree with each other where they intersect, it is clear that together they induce a map $\phi : \mathrm{colim}_I X \to Y$, such that $(I, \phi) \in S$ and is an upper bound for $\{(I_t, \phi_t)|t \in T\}$. Since $(A, g) \in S$, S is non-empty, and so Zorn's lemma tells us that S has a maximal element (I, ϕ). If $I \neq \lambda$, then it would have to correspond to a successor ordinal $\beta + 1$ due to the argument above. However, in that case we can extend ϕ by considering a lift in the diagram:

$$\begin{tikzcd} X_\beta \arrow[r, "\phi"] \arrow[d] & Y \arrow[d, "p"] \\ X_{\beta+1} \arrow[r, "h \circ i_{\beta+1}"'] \arrow[ur, dashed] & Z \end{tikzcd}$$

where $i_{\beta+1}$ is a leg of $\mathrm{colim}X$ viewed as a cocone. Therefore, $I = \lambda$ and so we have found a lift in our original commutative square. $\square$

4.2 Small objects

We now extend the notion of smallness in [5] to arbitrary cardinals:

Definition: Let D be a subcategory of C. If κ is a cardinal, an object $W \in C$ is said to be κ-small relative to D if for every regular cardinal $\lambda \geq \kappa$, and every λ-sequence $X : \lambda \to C$ in D, the map of sets $\mathrm{colim}_{\beta<\lambda} C(W, X_\beta) \to C(W, \mathrm{colim}_{\beta<\lambda} X_\beta)$ is an isomorphism. W is said to be small relative to D if it is κ-small relative to D, for some cardinal κ.

Note that if I is a set of objects that are small relative to D, then there is a cardinal κ such that every element of I is κ-small relative to D. The following lemma is a good exercise in the definition of smallness:

Lemma 4.2.1: *If J is a small category and $W : J \to C$ is a diagram in C such that W_i is small relative to D for every object $i \in J$, then $colim_J W$ is small relative to D.*

Proof: It is enough to prove this for coproducts and coequalisers. We'll write W_c for the colimit of W. In the case that W is a coproduct, let κ be a cardinal such that every W_i is κ-small relative to D and let λ be a regular cardinal greater than both κ and the cardinality of the set of objects of J. Then, if $X : \lambda \to C$ is a λ-sequence in D, and $\coprod g_i : W_c \to \mathrm{colim} X$ is a map, then every g_i factors through X_{β_i} for some $\beta_i < \lambda$. Since λ is a regular cardinal greater than the cardinality of W, it follows that there is some X_β such that every g_i factors through X_β, which proves surjectivity of the required map. For injectivity, suppose that $h : W_c \to X_\beta$ and $k : W_c \to X_{\beta'}$ are both factorisations of g. We may as well assume $\beta = \beta'$. Then for each $i \in W$ there exists $\beta_i \geq \beta$ such that $ih_i = ik_i$, where i is the map from X_β to X_{β_i}, since W_i is κ-small relative to D. Then the regularity of the cardinal λ again implies we can suppose that $\beta_i = \beta_j$ for all i, j and this proves injectivity.

In the case where W is a coequaliser, let λ be an infinite regular cardinal greater than κ, where κ is defined as in the coproduct case. Let $X : \lambda \to C$ be a λ-sequence in D and suppose that $g : W_c \to \mathrm{colim} X$ is a map. Then there is a factorisation of the map from W_2 through X_β for some β. Hence, we have two factorisations of the map from W_1, induced by g, through X_β, and so these become equal at X_γ for some $\gamma \geq \beta$, since W_1 is also κ-small. This proves surjectivity and injectivity is immediate from the κ-smallness of W_2. $\square$

Before turning to the small object argument we give an example of smallness in the category of all topological spaces. Recall that a relative cell complex is a transfinite composition of cell attachments. The following well-known topological fact will allow us to prove that all compact spaces are sequentially small relative to the subcategory of relative cell complexes.

Lemma 4.2.2: *If $i : A \to X$ is a relative cell complex, then any compact subspace K of X intersects the interior of only finitely many cells of $X - A$.*

Proof: We can express the map $A \to X$ as the transfinite composite of a λ-sequence, $X : \lambda \to \mathbf{Top}$, where each map $X_\beta \to X_{\beta+1}$ is a pushout of an inclusion of the form $S^{n-1} \to D^n$. Suppose that K is a compact subspace of X which intersects the interior of infinitely many cells of $X - A$. Then define β_n to be the minimal element of λ such that K intersects the interior of n-cells of $X_{\beta_n} - A$. Since we are only attaching one cell at a time, $\beta_n \neq \beta_{n+1}$ for any n. Let $X_\infty = \mathrm{colim}_n X_{\beta_n}$. It is staightforward to show that, for any β, X_β is a closed subspace of X, using transfinite induction and the definition of the colimit topology as a type of quotient. In particular, $K \cap X_\infty$ is compact. Let $x_i \in (X_{\beta_i} \setminus X_{\beta_{i-1}}) \cap K$, for $i \geq 1$. Then $\{x_i\}$ is closed in X, for all i. We have $X_\infty = \cup_{i=1}^{\infty}(X_\infty \setminus \{x_i, x_{i+1}, ...\})$, where each set in the union is open due to definition of the colimit topology. However, $K \cap X_\infty$ is not contained in any finite union of the sets in question, a contradiction, since $K \cap X_\infty$ is compact. $\square$

Corollary 4.2.3: *If K is a compact topological space, then K is $\aleph_0$-small relative to the subcategory of relative cell complexes, D.*

Proof: Suppose that $\lambda \geq \aleph_0$ is a regular cardinal and $X : \lambda \to \mathbf{Top}$ is a λ-sequence in D. If $g : K \to \mathrm{colim}X$ is a map, then let n denote the number of cells of $X - X_0$ that $g(K)$ intersects the interior of. Lemma 4.2.2 tells us that n is finite. Let β be the minimal element of λ such that $g(K)$ intersects the interior of n cells of $X_\beta - X_0$. Then g factors through X_β, since X_β is a subspace of $\mathrm{colim}X$. Moreover, any factorisation of g through X_γ, for some $\gamma < \lambda$, is unique since the map $X_\gamma \to \mathrm{colim}X$ is a monomorphism. $\square$

4.3 The small object argument

We now state and prove the small object argument for arbitrary cardinals κ. As we will see, the proof is a generalisation of the proof for the case $\kappa = \aleph_0$.

Definition: Let I be a set of maps. The subcategory of I-injectives is the subcategory of maps with the RLP with respect to all elements of I. The subcategory of I-cofibrations is the subcategory consisting of maps with the LLP with respect to all I-injectives.

Definition: The subcategory of relative I-cell complexes is the subcategory of maps which are a transfinite composition of pushouts of elements of I. If κ is a cardinal, then an object is defined to be κ-small relative to the set I if it is κ-small with respect to the subcategory of relative I-cell complexes.

Definition: We say that I permits the small object argument if the domains of all elements of I are small relative to I.

Theorem 4.3.1: *If I is a set of maps in C that permits the small object argument, then any map f in C is of the form pi, where i is a relative I-cell complex and p is an I-injective.*

Proof: Let $f : Y \to Z$ be a map. Suppose that λ is the minimal ordinal such that there is not a unique pair (X, g), where X is a λ-sequence and $g : \mathrm{colim}_{\beta<\lambda}(X) \to Z$ is a map such that $gi = f$, where i is the composition of X, and for every $\beta + 1 < \lambda$, there is a pushout square:

$$\begin{tikzcd}
\coprod_{j\in S_\beta} A_j \arrow[r, "\coprod h_j"] \arrow[d, "\coprod u_j"'] & \coprod_{j\in S_\beta} B_j \arrow[d] \\
X_\beta \arrow[r] & X_{\beta+1}
\end{tikzcd}$$

where S_β is the set of commutative squares of the form:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "h"'] & X_\beta \arrow[d, "gi_\beta"] \\
B \arrow[r, "v"'] & Z
\end{tikzcd}$$

Moreover, for each β with $\beta + 1 < \lambda$, $g_{\beta+1}$ is the map induced by g_β and $\coprod_{j\in S_\beta} v_j$.

Then λ cannot be a limit ordinal since then (X, g) could be defined, uniquely, as the colimit of (X_β, g_β) for $\beta < \lambda$. It is trivial that $\lambda \neq \beta + 1$ for a limit ordinal β. If $\lambda = \beta + 2$ for some β, then we can extend (X_β, g_β) by defining $X_\beta \to X_{\beta+1}$ to be the pushout as above for $X_\beta := \mathrm{colim}(X)$. Then, by the definition of S_β, $g_\beta u_j = v_j h_j$ for all $j \in S_\beta$ and so g_β and $\coprod_{j\in S_\beta} v_j$ define a map $g_{\beta+1} : X_{\beta+1} \to Z$ by the universal property of the pushout, satisfying $g_{\beta+1} i = f$. Moreover, it is obvious that g is unique due to it being unique up to g_β, and $g_{\beta+1}$ being induced by the universal property of the pushout.

Hence, no such minimal ordinal exists and so for any ordinal λ, there is a unique (X, g) satisfying the specified properties. In particular, we can take $\lambda = \kappa$ where every domain of I is κ-small relative to I and κ is an infinite regular cardinal. If $i : Y \to Q$ is the transfinite composite of X and $p = g$, then we have $f = pi$ and i is a relative I-cell complex. We claim that p is an I-injective. Suppose that we have a diagram:

$$\begin{tikzcd}
 & X_\beta \arrow[d, dashed, "i_\beta"] \\
A \arrow[ur, dashed, "u"] \arrow[r, "u'"] \arrow[d, "h"'] & Q \arrow[d, "p"] \\
B \arrow[r, "v"] & Z
\end{tikzcd}$$

where h is a map in I. Then, since A is κ-small, $u^{'}$ factors through some X_β for $\beta < \kappa$, as shown. Since infinite regular cardinals are limit ordinals, we have a pushout square:

$$\begin{tikzcd}
\coprod_{j \in S_\beta} A_j \arrow[r, "\coprod h_j"] \arrow[d, "\coprod u_j"'] & \coprod_{j \in S_\beta} B_j \arrow[d] \\
X_\beta \arrow[r] & X_{\beta+1}
\end{tikzcd}$$

where there exists $j \in S_\beta$ such that $u = u_j, v = v_j$ and $h = h_j$. Let ϕ be the jth factor of the map on the RHS. Then $i_{\beta+1}\phi$ is a lift of the solid square above, essentially by definition. Hence, p is an I-injective as required. $\square$

Note that the construction of the factorisation of f into pi given above is "functorial" and so model categories in which the small object argument can be used to find factorisations are functorial model categories. However, in this essay we have decided not to require that model categories are functorial.

5 Simplicial Sets

In this section, we will show that there is a model structure on the category of simplicial sets which is Quillen equivalent to the Quillen model structure on topological spaces. We will follow Chapter 1 of [3] closely, and will take the first 50 pages for granted, although we give a brief outline below. We then turn our attention to the theory of minimal fibrations which we'll use to show that the realisation functor takes Kan fibrations to Serre fibrations. In fact, we'll prove the stronger statement, due to Fritsch and Piccinini, that the realisation functor takes Kan fibrations to Hurewicz fibrations in the category of compactly generated spaces ([8], Theorem 4.5.25). After proving this, we will proceed with constructing the desired model structure on simplicial sets.

5.1 Preliminaries

We define a Kan fibration to be a map with the RLP with respect to the maps $\{\Lambda^n_k \to \Delta^n\}$, for all n and k. We call a fibrant simplicial set a Kan complex. For example, if X is a topological space, then SX is a Kan complex. We define an anodyne extension to be a member of the saturation of the set of monomorphisms $\{\Lambda^n_k \to \Delta^n\}$. Informally, anodyne extensions are maps which can be formed by taking pushouts, retracts and countable composites of elements of $\{\Lambda^n_k \to \Delta^n\}$. A key point is that if $K \subset L$ is an arbitrary inclusion of simplicial sets and $A \hookrightarrow B$ is an anodyne extension, then the inclusion $A \times L \cup B \times K \to B \times L$ is an anodyne extension and so has the LLP with respect to all Kan fibrations.

If X is any simplicial set then the functor $(-) \times X$ has a right adjoint denoted by $\mathrm{Hom}(X, -)$. The Hom functor preserves Kan fibrations and if $K \subset L$ is an inclusion of simplicial sets then, for any Kan complex X, the map $\mathrm{Hom}(L, X) \to \mathrm{Hom}(K, X)$ is a Kan fibration. If X is a Kan complex, then we can define the homotopy group $\pi_n(X, x)$, for any $n \in \mathbb{N}$ and $x \in X_0$, and we call a map between Kan complexes a weak equivalence if it induces an isomorphism on all homotopy groups for all choices of basepoint, as well as an isomorphism on π_0. If $p : X \to Y$ is a fibration between Kan complexes, then there is a corresponding long exact sequence of homotopy groups analogous to the one defined for topological spaces. We then have the following proposition:

Proposition 5.1.1: *Let $f : X \to Y$ be a map between Kan complexes. Then f is a fibration and a weak equivalence iff it has the RLP with respect to the maps $\partial\Delta^n \to \Delta^n$ for all n.*

Proving that f is a fibration and a weak equivalence if it has the RLP with respect to all inclusions of simplicial sets is straightforward and the reverse direction can be proved with a direct argument.

5.2 Minimal fibrations

Our first goal in this section is to prove that the realisation functor preserves fibrations and to do this it is conceptually nicer to work with the category of compactly generated spaces (see, for example, Ch 5 of [1]), **CGHaus**, rather than the category of all topological spaces. This is because the functor $|-|:$ **sSet** $\to$ **CGHaus** preserves finite limits, which will allow us to prove that certain maps arising as the geometric realisation of a Kan fibration are locally trivial. We begin with the following observation:

Definition: If $p : X \to Y$ is a Kan fibration, then there is an equivalence relation on the set X_n defined by $x \sim_p y$ if $\partial x = \partial y$ and there exists a fiberwise homotopy, rel $\partial\Delta^n$, between x and y. We say that x and y are p-related if they are in the same equivalence class (p-class).

Definition: We say that $p : X \to Y$ is a minimal fibration if it is a Kan fibration such that, for any n, any two p-related elements of X_n are equal.

Lemma 5.2.1: *If x and y are degenerate r-simplices of a simplicial set X such that $\partial x = \partial y$, then $x = y$.*

Proof: Any simplex can be written, uniquely, in the form $x = s_I(a)$ for some finite subset of the integers I and nondegenerate simplex a. If $\partial x = \partial y$, and $x = s_I(a), y = s_J(b)$, with I, J non-empty, then we have:

$$a = d_I(x) = d_I(y) = d_I s_J(b)$$

and, similarly, $b = d_J s_I(a)$. Together, these imply $I = J$ and $a = b$, and, so, $x = y$. $\square$

It is worth noting that the takeaway of the above lemma is that each p-class contains at most one degenerate simplex and not that the relation is trivial for degenerate simplices. In particular, a degenerate simplex can be p-related to a nondegenerate simplex. Lemma 5.2.1 allows us to prove:

Lemma 5.2.2: *If $p : X \to Y$ is a Kan fibration, then there is a strong fiberwise deformation retract $r : X \to Z$ such that $p : Z \to Y$ is a minimal fibration.*

Proof: Let $n \geq -1$ and suppose that we have defined a fiberwise homotopy $H_n : X \to X$ between f and a map r_n such that if $X^n := r_n(X)$, then:

i) if x and y are distinct elements of X^n_k, where $k \leq n$, then x and y are not p-related in X,

ii) H_n is the constant homotopy when restricted to X^n_n.

Recall that we have a pushout square:

$$\begin{array}{ccc}
\coprod_{\alpha\in S}\ \partial\Delta^{n+1} & \longrightarrow & X_n \\
\big\downarrow & & \big\downarrow \\
\coprod_{\alpha\in S}\ \Delta^{n+1} & \longrightarrow & X_{n+1}
\end{array}$$

where the disjoint union is indexed over the set S of nondegenerate elements of X_{n+1}. For each p-class β of X_{n+1} choose an element $x_\beta \in \beta$ which, if possible, is degenerate. Now, if $x \in S$, define a homotopy $K_x : \Delta^{n+1} \times \Delta^1 \to X$ starting from $r_n(x)$ as follows:

i) if $x = x_\beta$ for some x_β with $\partial x_\beta \subset X_n^n$, then let K_x be a fiberwise inverse, rel $\partial\Delta^{n+1}$, to $H_n \circ (x \times 1)$,

ii) otherwise, let K_x be a fiberwise homotopy, rel $\partial\Delta^{n+1}$, between $r_n(x)$ and x_β where β is the p-class containing $r_n(x)$.

Then, since $(-)\times\ \Delta^1$ is a left adjoint, and so preserves colimits, we have defined a fiberwise homotopy, rel X_n, between $r_n : X_{n+1} \to X$ and a map $s : X_{n+1} \to X$. We can extend this to a fiberwise homotopy L, rel X_n, between $r_n : X \to X$ and a map which we'll call $r_{n+1} : X \to X$. Let $X^{n+1} = r_{n+1}(X)$. By construction, $X_k^{n+1} = X_k^n$ for $k \leq n$ and the non-degenerate elements of X_{n+1}^{n+1} are the chosen representatives of their p-class in X and so any two distinct such elements are not p-related in X, nor are they p-related to any degenerate simplices.

We have a diagram:

$$\begin{tikzcd}
X \times \Lambda_1^2 \cup X_{n+1}^{n+1} \times \Delta^2 \cup X_n \times \Delta^2 \arrow[rr, "H_n \cup L \cup T \cup H_n \circ (1\times c)"] \arrow[d] & & X \arrow[d, "p"] \\
X \times \Delta^2 \arrow[rru, dashed, "K"] \arrow[rr, "p\circ\pi_X"'] & & Y
\end{tikzcd}$$

where T is defined by the fiberwise inverses we choose in step i) of constructing the homotopies K_x above and c is induced by the order-preserving map $\mathbf{2}$ to $\mathbf{1}$ sending 0 to 0, 1 to 1 and 2 to 1. Now, let H_{n+1} be the restriction of K to the 1st face of Δ^2. Then H_{n+1} is the constant homotopy on X_{n+1}^{n+1} by construction and, moreover, $H_{n+1}|_{X_n\times\Delta^1} = H_n|_{X_n\times\Delta^1}$. Therefore, defining $Z = colim_n X_n^n$ and H to be the colimit of the H_n, we finish the proof. $\square$

The next lemma, intuitively, tells us that the retraction defined above in Lemma 5.5.2 is an "acyclic fibration", although, so far, we have only defined weak equivalences between Kan complexes, so we cannot yet call it acyclic.

Lemma 5.2.3: *Let $p : X \to Y$ be a fibration. Then, if $f : X \to Z$ is a strong fiberwise deformation retract such that $p : Z \to Y$ is a minimal fibration, f has the RLP with respect to all of the maps $\partial\Delta^n \to \Delta^n$.*

Proof: Suppose that we have a diagram:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "z"'] & Z
\end{tikzcd}$$

Then we have a diagram:

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{1\} \arrow[r, "H\circ(g\times 1)\cup z"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "pz\circ\pi_{\partial\Delta^n}"'] \arrow[ur, dashed, "K"] & Y
\end{tikzcd}$$

and another:

$$\begin{tikzcd}
\Delta^n \times \Lambda^2_0 \cup \partial\Delta^n \times \Delta^2 \arrow[r, "K\cup H\circ(K_0\times 1)\cup H\circ(g\times 1)\circ(1\times c)"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^2 \arrow[r, "pz\circ\pi_{\Delta^n}"'] \arrow[ur, dashed, "L"] & Y
\end{tikzcd}$$

where $c : \Delta^2 \to \Delta^1$ is induced by the order-preserving map sending 0 to 0, 1 to 1 and 2 to 1. Then, the restriction of fL to the 0th face gives a fiberwise homotopy rel $\partial\Delta^n$ between z and fK_0 and so the minimality of p restricted to Z shows us that $fK_0 = z$ and, therefore, that K_0 is a lift for our original diagram. □

We've seen that any Kan fibration strongly deformation retracts onto a minimal fibration. We'll now show that minimal fibrations are locally trivial which, in turn, will allow us to prove that the realisation of a minimal fibration is a Hurewicz fibration. A key observation is:

Lemma 5.2.4: *Suppose that we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "q"'] & & Y \arrow[dl, "p"] \\ & Z & \end{tikzcd}$$

where p is a minimal fibration, g is an isomorphism, and f and g are fiberwise homotopic. Then f is an isomorphism.

Proof: By precomposing with g^{-1}, we may assume that $X = Y$ and $g = 1$. Let H be a fiberwise homotopy from f to 1. Suppose that f is an isomorphism on X_k for each $k < n$. Let $x : \Delta^n \to X$ be an element of X_n. Then, $\partial x = fv$ for some unique map $v : \partial\Delta^n \to X$. We have a diagram:

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \arrow[r, "Hv \cup x"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r, "px \circ \pi_{\Delta^n}"'] \arrow[ur, dashed, "K"] & Z \end{tikzcd}$$

Then, since p is minimal, $\phi(x) := K|_{\Delta^n \times \{1\}}$ is unique and so we have defined an operation on X_n which sends x to $\phi(x)$. Note that, if $x = f(y)$ for some y, then $\phi(x) = y$.

Moreover, we have a diagram:

$$\begin{tikzcd} \Delta^n \times \Lambda^2_2 \cup \partial\Delta^n \times \Delta^2 \arrow[r, "K \cup H\phi(x) \cup H\circ(1\times c)"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \times \Delta^2 \arrow[r, "px \circ \pi_{\Delta^n}"'] \arrow[ur, dashed] & Z \end{tikzcd}$$

where c is induced by the order preserving map sending 0 to 0, 1 to 0 and 2 to 1. This shows that $f\phi(x)$ is fiberwise homotopic to x, rel $\partial\Delta^n$, and so, by the minimality of p, we have $f\phi(x) = x$, which shows that f is an isomorphism on X_n. $\square$

Corollary 5.2.5: *Suppose that we have a commutative triangle:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \arrow[dl, "p"] \\ & Z & \end{tikzcd}$$

where p, q are minimal fibrations and f is a fiberwise homotopy equivalence. Then f is an isomorphism.

To conclude our proof of the local triviality of minimal fibrations, we will need the following result concerning pullbacks of fibrations along homotopic maps.

Lemma 5.2.6: *If we have two pullback diagrams, $i = 0, 1$:*

$$\begin{tikzcd} P_i \arrow[r] \arrow[d, "q_i"'] & X \arrow[d, "p"] \\ A \arrow[r, "f_i"'] & Y \end{tikzcd}$$

where p is a Kan fibration and f_0, f_1 are homotopic via a homotopy H, then P_0, P_1 are fiberwise homotopy equivalent (over A).

Proof: We have a diagram:

$$\begin{tikzcd} P_0 \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ P_0 \times \Delta^1 \arrow[r, "H \circ (q_0 \times 1)"'] \arrow[ur, dashed, "K"'] & Y \end{tikzcd}$$

and K_1 together with q_0 defines a map $\phi : P_0 \to P_1$. Similarly, we can define a map $\psi : P_1 \to P_0$ by defining an analogous homotopy $L : P_1 \times \Delta^1 \to X$.

We then have a diagram:

$$\begin{tikzcd} P_0 \times \Lambda^2_2 \arrow[r, "L \circ (\phi \times 1) \cup K"] \arrow[d] & X \arrow[d, "p"] \\ P_0 \times \Delta^2 \arrow[r, "H \circ (1 \times c) \circ (q_0 \times 1)"'] \arrow[ur, dashed, "M"'] & Y \end{tikzcd}$$

where c is induced from an order-preserving map. Then, if N is the restriction of M to the 2nd face, we have a diagram:

$$\begin{tikzcd} P_0 \times \Delta^1 \arrow[r, "N"] \arrow[d, "q_0 \circ \pi"'] & X \arrow[d, "p"] \\ A \arrow[r, "f_0"'] & Y \end{tikzcd}$$

inducing a fiberwise homotopy between 1 and $\psi\phi$. Similarly, $\phi\psi$ is fiberwise homotopic to 1. $\square$

Corollary 5.2.7: *If we have a pullback square:*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "q"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "y"'] & Y \end{tikzcd}$$

where Y is connected with basepoint $\star \in Y_0$, and p is a minimal fibration, then P is isomorphic over Δ^n to $F \times \Delta^n$, where F is the fiber of p over the basepoint of Y.

Proof: First of all, there is a homotopy from y to the constant map into $y(0)$. Then, since Y is connected, there is a chain of homotopies which relate $y(0)$ and $\star$ and these specify a chain of fiberwise isomorphisms between P and $F \times \Delta^n$. $\square$

5.3 Realisations of fibrations

Theorem 5.3.1: *If $p : X \to Y$ is a minimal fibration, then $|p| : |X| \to |Y|$ is a Hurewicz fibration.*

Proof: We'll prove inductively that $|p|$ is locally trivial, in the sense that there is a cover of $|Y|$ by closed sets, whose interiors cover $|Y|$, such that $|p|$ is trivial with fibre $|F|$ over each of the closed sets. We'll first prove that $|p|$ is a Serre fibration. Since disks are compact, it suffices to prove that $|p|$ remains locally trivial when we add a cell.

Suppose that A is a subset of Y such that $|p|$ is locally trivial, in the sense descibed above, when restricted to $|p|^{-1}(|A|)$, and B is obtained from A by attaching a nondegenerate simplex β. From this point onwards we will drop the $|-|$ notation and work only with spaces and not simplicial sets. Let D be a closed subset of A over which p is trivial and let E be the closed neighbourhood $(\partial\beta \cap D) \times [0, \frac{1}{2}]$ of D in B which strongly deformation retracts onto D via the obvious choice of map $r : E \to D$. Here $\partial\beta \times [0, \frac{1}{2}]$ denotes the image in B of a boundary annulus of the cell β of half the radius of the whole cell. Let $D^{'} = \beta^{-1}(D), E^{'} = \beta^{-1}(E) \subset |\Delta^n|$. Then $E^{'} \to \beta(E^{'})$ is a proper map that is also a quotient map.

Observe that $D^{'} \times_p X$ is a subspace of $E^{'} \times_p X$. By Corollary 5.2.7, we have a fiberwise homeomorphism:

$$\begin{tikzcd} E^{'} \times F \arrow[rr, "\delta"] \arrow[dr] & & E^{'} \times_p X \arrow[dl] \\ & E^{'} & \end{tikzcd}$$

which restricts to a fiberwise homeomorphism $\delta : D^{'} \times F \to D^{'} \times_p X$ over $D^{'}$. By assumption, we also have a fiberwise homeomorphism $\omega : D \times F \to D \times_p X = p^{-1}(D)$ which pulls back to a fiberwise homeomorphism:

$$\begin{tikzcd} D^{'} \times F \arrow[rr, "\omega^{'}"] \arrow[dr] & & D^{'} \times_p X \arrow[dl] \\ & D^{'} & \end{tikzcd}$$

Next we will modify δ so that it restricts to $\omega^{'}$ on $D^{'} \times F$. We have a composite fiberwise homeomorphism $\delta^{-1}\omega^{'} : D^{'} \times F \to D^{'} \times F$ and we can extend it to a fiberwise homeomorphism $\phi : E^{'} \times F \to E^{'} \times F$ by the formula $\phi(e, x) = (e, \pi_F \delta^{-1}\omega^{'}(r(e), x))$. Then, letting $\tau = \delta\phi$, we have that τ restricted to $D^{'} \times F$ is equal to $\omega^{'}$.

Using the exponential law, or otherwise, observe that the map $E^{'} \times F \to \beta(E^{'}) \times F$ is also a quotient map, and so the fact that $\tau = \omega^{'}$ on $D^{'} \times F$ and the universal property of the quotient implies we have a continuous bijection k:

$$\begin{array}{ccc} E^{'} \times F & \xrightarrow{\tau} & E^{'} \times_p X \\ \downarrow & & \downarrow \\ \beta(E^{'}) \times F & \overset{k}{\dashrightarrow} & \beta(E^{'}) \times_p X \end{array}$$

The map on the right is proper (ie the preimages of compact sets are compact) since it is the pullback of the proper map $E^{'} \to \beta(E)$ along a map into a Hausdorff space. Hence, k is proper and note that k restricts to a homeomorphism onto its image for any compact subspace of $\beta(E^{'}) \times F$ since $\beta(E^{'}) \times_p X$ is Hausdorff. It is an elementary fact (see, for example, page 39 of [1]) that a map of sets $\psi : M \to N$ between two compactly generated spaces is continuous iff its restriction to every compact subspace is continuous. It follows that k is a fiberwise homeomorphism. Note also that $k = \omega$ on $\beta(D^{'}) \times F$ and so we can glue k and ω together to obtain a fiberwise homeomorphism of $E \times F$ onto $p^{-1}(E)$. Using these closed sets E for every closed set D in the assumed cover of A, as well as a closed interior disc of the cell β, we have found the required cover of B.

To show that p is a Hurewicz fibration, we now consider the case where $A = Y_n$ and $B = Y_{n+1}$ are successive skeletons of Y. Again, let D be a closed subspace over which p is trivial and let $\omega : D \times F \to D \times_p X$ be a fiberwise homeomorphism. For each $(n+1)$-cell β of Y, let E_β be the closed set which we would call E if we were just adding the single cell β, as above. Let E be the union of the E_β. We have fiberwise homeomorphisms $\tau_\beta : E_\beta \times F \to E_\beta \times_p X$ which restrict to ω on $D \times F$. Hence, they glue together to give an isomorphism of sets $E \times F \to E \times_p X$. Both $E \times F$ and $E \times_p X$ are compactly generated, so to show that this map is a homeomorphism we only need to consider compact subsets K of $E \times F$ and $E \times_p X = p^{-1}(E)$. However, any compact subset can only intersect finitely many of the sets $E_\beta \times F$ or $E_\beta \times_p X$, and so continuity in both directions follows from the gluing lemma for continuous maps defined on finitely many closed sets. Therefore, taking the subspace E induced by each D in the closed cover of Y_n, as well as interior closed disks of each $(n+1)$-cell, we get an induced closed cover of Y_{n+1} satisfying the desired properties.

Finally, to show that p is locally trivial, let $y \in Y_n$ and let D be a closed neighbourhood of y in Y_n over which $p|_{p^{-1}(Y_n)}$ is locally trivial. Let $D_0 = D$ and define D_1 to be the closed subset E of Y_{n+1} as defined in the previous paragraph. Inductively, let D_{i+1} be the $E \subset Y_{n+i}$ constructed from D_i. Then we have homeomorphisms $\omega_i : D_i \times F \to D_i \times_p X$ which glue together to form a bijection of sets $\omega : E \times F \to E \times_p X$, where E is the union of the D_i. An identical compactness argument then tells us that ω is a homeomorphism and, since there is an open subset of Y contained in E, containing y, over which p is trivial, we can conclude that p is a Hurewicz fibration, as desired. □

Lemma 5.3.2: *If $f : X \to Y$ has the RLP with respect to all of the maps $\partial\Delta^n \to \Delta^n$, then $|f|$ is a Hurewicz fibration.*

Proof: Since f has the RLP with respect to all inclusions of simplicial sets, it is enough to show that f can be factored as the composite of an inclusion and a map g such that $|g|$ is a Hurewicz fibration, by the retract argument. Explicitly,

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1\times f"'] & X \arrow[d, "f"] \\ X \times Y \arrow[r, "\pi_Y"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

shows us that f is a retract of π_Y and, since $|-|$ preserves finite limits, $|\pi_Y| = \pi_{|Y|}$, and so is a Hurewicz fibration. Therefore, so is $|f|$. $\square$

Given a Kan fibration p, there is a strong deformation retract r of p onto a minimal fibration, and r has the RLP with respect to all the maps $\partial\Delta^n \to \Delta^n$, by Lemma 5.2.3. We've seen that the realisation of a minimal fibration is a Hurewicz fibration, and we've just seen the same is true of the realisation of r. Putting everything together, we can now conclude:

Theorem 5.3.3: *If $p : X \to Y$ is a Kan fibration, then $|p|$ is a Hurewicz fibration.*

From this point onwards, we will work in the category of all topological spaces, **Top**, rather than **CGHaus**, and Theorem 5.3.3 has the following consequence:

Theorem 5.3.4: *If $p : X \to Y$ is a Kan fibration, then $|p|$ is a Serre fibration.*

5.4 Proof of the model axioms

At first glance, it is somewhat surprising that we have already recovered enough topological information to prove the next result. For example, the theorem implies that the homotopy groups of a Kan complex X are isomorphic to the homotopy groups of its realisation $|X|$, yet we have not mentioned any kind of simplicial approximation theorem for maps into $|X|$. For motivation, one can consider the fibrant simplicial set BG, for a discrete group G. It is an easy exercise to show that $\pi_1(BG) \cong G$ and $\pi_i(BG) = 0$ for $i \geq 2$. To calculate the homotopy groups of $|BG|$, one can consider a fibration $|EG| \to |BG|$, where the space $|EG|$ is contractible. Since we have already proved that $|-|$ preserves fibrations, it turns out that we can use the same idea to prove:

Theorem 5.4.1: *If X is a Kan complex, then the unit $\eta_X : X \to S|X|$ is a weak equivalence.*

Proof: Let $x \in X_0$. Then, by the small object argument, we can factor $x : \Delta^0 \to X$ as $x = pi$, where p is a Kan fibration and $i : \Delta^0 \to Z$ is an anodyne extension. The class of anodyne extensions which induce a weak equivalence on passage to realisation is saturated and this implies that $\pi_i(|Z|) \cong 0$ for all i. Now consider the diagram:

$$\begin{tikzcd}
\Delta^0 \arrow[r, "c"] \arrow[d] & \mathrm{Hom}(\Delta^1, Z) \arrow[d] \\
Z \arrow[r, "1\times\star"'] \arrow[ur, dashed, "H"] & Z \times Z
\end{tikzcd}$$

where c is the constant homotopy determined by the basepoint. The right hand map is induced by the inclusion $\partial\Delta^1 \to \Delta^1$ and is a fibration since Z is a Kan complex. Therefore, the lift defines a homotopy between 1 and the constant map rel Δ^0 and so $\pi_i(Z) \cong 0$ for all i.

Now consider the commutative square:

$$\begin{tikzcd}
\pi_i(X) \arrow[r, "\cong"] \arrow[d, "\eta_X"'] & \pi_{i-1}(F) \arrow[d, "\eta_F"] \\
\pi_i(S|X|) \arrow[r, "\cong"'] & \pi_{i-1}(S|F|)
\end{tikzcd}$$

It is easy to see that η_X determines an isomorphism on 0-simplices for any Kan complex X and, therefore, an inductive argument using the above constructions and commutative square gives the result. $\square$

Corollary 5.4.2: *For all topological spaces X, the counit $\epsilon_X : |SX| \to X$ is a weak equivalence.*

Proof: The functor S creates weak equivalences and, so, $S\epsilon_X \circ \eta_{SX} = 1$ implies that ϵ_X is a weak equivalence by the two out of three property for weak equivalences (between Kan complexes) proved earlier on in [3].

$\square$

In light of Theorem 5.4.1, we can now extend our definition of a weak equivalence between Kan complexes to maps between arbitrary simplicial sets:

Definition: A map $f : X \to Y$ between simplicial sets is called a weak equivalence if $|f|$ is a weak equivalence.

We can also extend Proposition 5.1.1 to:

Proposition 5.4.3: *A map $p : X \to Y$ is a Kan fibration and a weak equivalence iff it has the RLP with respect to all the maps $\partial\Delta^n \to \Delta^n$.*

Proof: First suppose that p has the RLP with respect to all inclusions of simplicial sets. Then, p is certainly a Kan fibration and we can consider the diagrams:

$$\begin{tikzcd} \star \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "1"'] \arrow[ur, dashed, "g"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \times \partial\Delta^1 \arrow[r, "{(gp,1)}"] \arrow[d] & X \arrow[d, "p"] \\ X \times \Delta^1 \arrow[r, "p\circ\pi_X"'] \arrow[ur, dashed, "H"'] & Y \end{tikzcd}$$

which show that p has a right inverse, g, and there is a fiberwise homotopy between gp and 1. Therefore, we have a retract diagram:

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "p"'] & X \times \Delta^1 \arrow[r, "\pi_X"] \arrow[d, "H"] & X \arrow[d, "p"] \\ Y \arrow[r, "g"'] & X \arrow[r, "p"'] & Y \end{tikzcd}$$

and, since $|H| \circ |i_1| = 1$, $|H|$ is a weak equivalence and, therefore, so is p.

Next, suppose that p is a Kan fibration that is also a weak equivalence. The same direct argument used in the proof of Proposition 5.1.1 reduces the problem to finding a lift in diagrams of the form:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "g"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "f"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

where both f and $g|_{\Lambda^n_0}$ are constant maps. Since $|p|$ is a weak equivalence, the fiber F is a Kan complex with vanishing homotopy groups. In particular, the restriction of g to the 0th face is nullhomotopic and so there exists an extension of g to Δ^n, with image in F, as required (Chapter I, Lemma 7.4 of [3]). $\square$

We can now prove:

Theorem 5.4.4: *There is a proper model structure on* ***sSet*** *where:*

i) the cofibrations are the inclusions of simplicial sets,

ii) the fibrations are the Kan fibrations,

iii) the weak equivalences are the maps whose geometric realisation is a weak equivalence of topological spaces.

Moreover, the adjunction $|-| \dashv S$ is a Quillen equivalence between ***Top****, with the q-model structure, and* ***sSet****.*

Proof: **MC1-3** are obvious and we've proved one half of **MC4** in Proposition 5.4.3. For **MC5**, we can use the small object argument to factor any map as an anodyne extension followed by a Kan fibration. Since the class of anodyne extensions whose realisation is a weak equivalence is saturated, we can conclude that all anodyne extensions are acyclic cofibrations. The second part of **MC5** follows directly from the small object argument and Proposition 5.4.3 above. For the second part of **MC4**, we can factor any acyclic cofibration, i, as an anodyne extension followed by an acyclic fibration (which has the RLP with respect to i). Therefore, the retract argument implies that i is itself an anodyne extension and so has the LLP with respect to any fibration. This completes the proof of the model axioms. Note, also, that anodyne extensions are equivalent to acyclic cofibrations.

The fact that **sSet** is left proper follows from the fact that all objects are cofibrant. To see that **sSet** is right proper we will use the fact that the Quillen model structure on **Top** is right proper (see Theorem 6.1.1). It follows that **sSet** is right proper, since $|-|$ preserves limits and fibrations and creates weak equivalences.

Finally, $(|-|, S)$ is a Quillen pair since S preserves fibrations and acyclic fibrations. The fact that it is a Quillen equivalence follows from the fact that $|-|$ creates the weak equivalences in **sSet**, by defintion, and ϵ_Y is a weak equivalence for all fibrant Y (see Lemma 2.2.2). □

6 Model Structures on Top

In this section, we will derive three model structures on **Top** and show that they are proper. The first is the already familiar Quillen model structure, which, as demonstrated in Theorem 5.4.4, is Quillen equivalent to the Quillen model structure on simplicial sets. The second model structure that we'll discuss is the Hurewicz model structure, in which the weak equivalences are the homotopy equivalences. Finally, we show that we can mix the Quillen and Hurewicz model structures to create a third model structure on **Top**, known as the mixed model structure. Mixed model structures, in general, were introduced relatively recently in a 2006 paper of M. Cole ([7]), which our treatment ultimately derives from (via [2]). There are also analogous q,h and m model structures on certain categories of chain complexes, but we will not discuss them in this essay.

6.1 The q-model structure

The q-model structure is the name that we will give to the Quillen model structure as described in [5]. Recall that we have:

Theorem 6.1.1: *There is a proper model structure on **Top** where:*

i) the weak equivalences are the traditional weak equivalences of topological spaces,

ii) the cofibrations are retracts of relative cell complexes,

iii) the fibrations are the Serre fibrations.

Since all objects are fibrant, Corollary 2.3.2 implies that the q-model structure is right proper. The fact that it is also left proper is a consequence of the gluing lemma below. In fact, it can be shown that, in any model category, left properness is equivalent to the (left) gluing lemma, but we will not need that result in this essay.

Lemma 6.1.2: *If i and j are q-cofibrations, and f, g, h are q-equivalences,*

$$\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[l, "i"'] \arrow[r, "k"] \arrow[d, "g"] & B \arrow[d, "h"] \\ A' & C' \arrow[l, "j"] \arrow[r, "l"'] & B' \end{tikzcd}$$

then the induced map between pushouts is a weak equivalence.

Proof: Since i and j are Hurewicz cofibrations we can replace each pushout with a double mapping cylinder. The result then follows from the fact that any map between excisive triads which restricts to a weak equivalence on the components of the triad and their intersection is a weak equivalence. $\square$

To see that this implies left properness, take $f, g, k = 1, i = j$, and $h = l$.

6.2 The h-model structure

We'll now derive the Hurewicz (or h) model structure on **Top**, which was first proved by Strøm in his aptly named paper, '*The homotopy category is a homotopy category*' ([11]).

Theorem 6.2.1: *There is a proper model structure on **Top** where:*

i) the weak equivalences are the homotopy equivalences,

ii) the cofibrations are the closed Hurewicz cofibrations,

iii) the fibrations are the Hurewicz fibrations.

MC1-3 are clear. For **MC4** we will need to use the characterisation of h-cofibrations as NDR-pairs.

Lemma 6.2.2: *Suppose that we have a commutative square, where i is an h-cofibration and p is an h-fibration:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "f"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Then a lift exists in the above diagram if either i or p is a homotopy equivalence.

Proof: First suppose that i is a homotopy equivalence. Then, since i is also a cofibration, we can find an $r : B \to A$ such that $ri = 1$ and $ir \simeq 1$ rel A. Then $pgr = fir \simeq f$ rel A via H, for some homotopy H. Let (K, u) represent (B, A) as an NDR-pair. Then, since H is a homotopy rel A, we can rescale H so that $H(b, t) = H(b, 1)$ for all $t \geq u(b)$. Now let L be a lift of H to X starting at gr. Define $\phi(b) = L(b, u(b))$. Then, due to the rescaling of H, $p\phi = f$ and, since $u(b) = 0$ iff $b \in A$, $\phi i = g$, as required.

Next, suppose that p is a homotopy equivalence. Then, since p is a fibration, there exists a map $j : Y \to X$ such that $pj = 1$ and $jp \simeq 1$ over Y. We can form the diagram:

$$\begin{tikzcd} A \times I \cup B \times \{0\} \arrow[r, "H \cup jf"] \arrow[d] & X \arrow[d, "p"] \\ B \times I \arrow[r, "f\pi_B"'] \arrow[ur, dashed, "K"] & Y \end{tikzcd}$$

where H is a homotopy $jfi = jpg \simeq g$ over Y, and the lift exists by the lifting axiom we've already proved. Then, $K|_{B\times\{1\}}$ gives us our desired lift. $\square$

To prove **MC5**, first recall that we can decompose any map into a cofibration followed by a homotopy equivalence using the mapping cylinder construction. Similarly, we can decompose any map into a homotopy equivalence followed by a fibration using the mapping path space construction. We will modify the second of these constructions to show that we can decompose any map into an h-acyclic cofibration followed by a fibration. The second part of **MC5** then follows immediately from the mapping cylinder construction.

Lemma 6.2.3: *If $f : X \to Y$ is a map between topological spaces, then f is of the form pi where i is an h-acyclic cofibration and p is a fibration.*

Proof: Define Qf to be the subspace of $X \times Y^I \times I$ consisting of triples (x, γ, t) such that $\gamma(0) = f(x)$ and $\gamma(s) = \gamma(t)$ for all $s \geq t$. Define a map $p : Qf \to Y$ by $p(x, \gamma, t) = \gamma(1)$. Define a map $i : X \to Qf$ by $i(x) = (x, c_{f(x)}, 0)$. Then, $f = pi$ so it is enough to show that i is an h-acyclic cofibration and p is a fibration.

In the case of i, consider the homotopy $H : Qf \times I \to Qf$ defined by:

$$H((x, \gamma, s), t) = (x, \gamma_{1-t}, \min(s, 1-t)),$$

where γ_{1-t} denotes the restriction of γ to $[0, 1-t]$ extended to a path from the unit interval by $\gamma_{1-t}(s) = \gamma(1-t)$ for $s \geq 1-t$. Then H defines a strong deformation retract of Qf onto $i(X)$ and, since $i(X) = u^{-1}(0)$, where $u(x, \gamma, t) = t$, it follows that i is an h-acyclic cofibration.

To see that p is a fibration, we will construct a path-lifting function $\lambda : Qf \times_p Y^I \to Qf^I$ by defining its adjoint $\tilde{\lambda} : Qf \times_p Y^I \times I \to Qf$ by:

$$\tilde{\lambda}((x, \gamma, t), \tau, s) = (x, \mu_{s,t}, \min(t+s, 1))$$

where, if $t + s \leq 1$:

$$\mu_{s,t}(a) = \begin{cases} \gamma(a) & \text{if } a \leq t \\ \tau(a-t) & \text{if } t \leq a \leq t+s \\ \tau(s) & \text{if } a \geq t+s \end{cases}$$

and, if $t + s \geq 1$:

$$\mu_{s,t}(a) = \begin{cases} \gamma(a(t+s)) & \text{if } a \leq \frac{t}{t+s} \\ \tau((a-1)t + as) & \text{if } \frac{t}{t+s} \leq a \leq 1 \end{cases}$$

□

For another direct proof of Lemma 6.2.3, see Strøm's original proof in [11], Proposition 2.

This completes our derivation of the h-model structure. Observe that it is proper since all objects are bifibrant.

6.3 The m-model structure

To finish this essay, we will demonstrate how we can combine the q and h-model structures to create a third model structure on **Top**, as described in the following theorem:

Theorem 6.3.1: *There is a proper model structure on **Top** where:*

i) the weak equivalences are the weak equivalences of topological spaces (the q-equivalences),

ii) the cofibrations are the h-cofibrations of the form fi where i is a q-cofibration and f is an h-equivalence,

iii) the fibrations are the Hurewicz fibrations.

We will call this model structure the m-model structure, where m stands for mixed.

We will work in a more general context. Suppose that M is a category with two model structures $(\mathcal{W}_q, \mathcal{C}_q, \mathcal{F}_q)$ and $(\mathcal{W}_h, \mathcal{C}_h, \mathcal{F}_h)$ such that $\mathcal{W}_h \subset \mathcal{W}_q$ and $\mathcal{F}_h \subset \mathcal{F}_q$ (and, so, $\mathcal{C}_q \subset \mathcal{C}_h$). The first thing we will prove is that the mixed model structure exists:

Lemma 6.3.2: *There is a model structure on M where:*

i) the weak equivalences are the q-equivalences,

ii) the fibrations are the h-fibrations.

Proof: Define $\mathcal{C}_m :={}^{\boxslash} (\mathcal{W}_m \cap \mathcal{F}_m) ={}^{\boxslash} (\mathcal{W}_q \cap \mathcal{F}_h)$. **MC1-3** are clear. One half of **MC4** is a definition.

For **MC5**, first factor $f = pi$ where p is an h-fibration and i is an h-acyclic h-cofibration. Then p is an m-fibration and i has the LLP with respect to $\mathcal{F}_h$ and, hence, with respect to $\mathcal{F}_h \cap \mathcal{W}_q$. It is also a q-equivalence since it is an h-equivalence.

For the second half of **MC5**, first factor $f = gi$ where i is a q-cofibration and g is a q-equivalence. Note that i is an m-cofibration. Next factor g as $g = pj$ where j is an m-acyclic m-cofibration and p is an m-fibration. Then, by the two out of three property, p is an m-acyclic m-fibration and so $p(ji)$ is our desired factorisation.

For the other half of **MC4**, it is enough to show that an m-acyclic m-cofibration i is an h-acyclic h-cofibration. Factor i as pj where p is an h-fibration and j is an h-acyclic h-cofibration. Since i is m-acyclic, p is m-acyclic. Since i is an m-cofibration it has the LLP with respect to p. Hence, the retract argument tells us that i is a retract of j and so an h-acyclic h-cofibration itself. $\square$

With the model structure demonstrated, Ken Brown's lemma has the following consequence:

Lemma 6.3.3: *Suppose that i and j are m-cofibrations in the following diagram:*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

i) if f is a q-equivalence, then it is an h-equivalence,

ii) if f is an h-cofibration, then it is an m-cofibration.

Proof: i) Consider the under-category $(A \downarrow M)$. Since f is an m-equivalence between m-cofibrant objects we can factor f as pi where i is an m-acyclic m-cofibration and p has a right inverse which is an m-acyclic m-cofibration. Now m-acyclic m-cofibrations are equivalent to h-acyclic h-cofibrations and, so, f is an h-equivalence.

ii) Factor f as pi where i is an m-cofibration and p is an m-acyclic m-fibration, By i), p is an h-acyclic h-fibration and so has the RLP with respect to f. Hence, the retract argument applies to show that f is an m-cofibration. □

We can use Lemma 6.3.3 to characterise the m-cofibrations:

Lemma 6.3.4: *A map $j : A \to X$ is an m-cofibration iff j is an h-cofibration which can be factored as fi, where i is a q-cofibration and f is an h-equivalence.*

Proof: Suppose that j is an m-cofibration. Clearly, it is also an h-cofibration. Factor j as pi where i is a q-cofibration and p is a q-acyclic q-fibration. Then, since all q-cofibrations are m-cofibrations, Lemma 6.3.3 applies to show that p is an h-equivalence, so we're done.

On the other hand, suppose that j is an h-cofibration which can be factored as fi where i is a q-cofibration and f is an h-equivalence. Factor f as pk, where p is an h-acyclic h-fibration and k is an h-acyclic h-cofibration. Note that i and k are both m-cofibrations and p has the RLP with respect to j. Therefore, the retract argument shows that j is an m-cofibration. □

Observe that the previous proof shows us that all m-cofibrations are retracts of a composite of a q-cofibration followed by an h-acyclic h-cofibration, and, when drawn out, the top row of the retract diagram consists of identity maps. We can use this to prove:

Lemma 6.3.5: *If M is right q-proper, then it is right m-proper. M is left m-proper iff it is left q-proper.*

Proof: The first sentence is obvious, as is the statement that left m-properness implies left q-properness. Therefore, suppose that M is left q-proper and that we have a pushout square where f is an m-equivalence and i is an m-cofibration:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & Y \end{array}$$

Then, by the comment above, g is a retract of the pushout of f along a composite of an h-acyclic h-cofibration and a q-cofibration. The pushout along either is an m-equivalence, since M is left q-proper, and so g is itself an m-equivalence. $\square$

This completes the proof of Theorem 6.3.1.

7 Bibliography

[1] - J. P. May. *A concise course in Algebraic Topology.* Chicago lectures in mathematics. University of Chicago Press, Chicago, Ill. 1999.

[2] - J. P. May and K. Ponto. *More concise Algebraic Topology: localization, completion, and model categories.* Chicago lectures in mathematics. University of Chicago Press, Chicago, Ill. 2012

[3] - P. Goerss and J. Jardine. *Simplicial homotopy theory*, Volume 174 of Progress in Mathematics. Birkhauser Verlag, Basel, 1999.

[4] - J. Dugundji. *Topology.* Allyn and Bacon, Boston, 1966.

[5] - W. G. Dwyer and J. Spalinski. *Homotopy theory and model categories.* Handbook of Algebraic Topology (I. M. James, ed.). Elsevier Science B. V., 1995.

[6] - P. S. Hirschhorn. *Model categories and their localizations*, Volume 99 of *Mathematical surveys and monographs.* American Mathematical Society, Providence, R. I., 2003.

[7] - M. Cole. *Mixing Model Structures.* Topology Appl., 153(7):1016-1032, 2006.

[8] - R. Fritsch and R. A. Piccinini. *Cellular structures in topology*, volume 19 of *Cambridge studies in advanced mathematics.* Cambridge University Press, Cambridge, 1990.

[9] - A. Joyal and M. Tierney. *Quasi-categories vs Segal spaces.* Categories in algebra, geometry and mathematical physics, 277-326. Contemp. Math., 431. Amer. Math. Soc., Providence, 2007.

[10] - D. Dugger. *Replacing model categories with simplicial ones*, Trans. Amer. Math. Soc. 353 (2001), no. 12, 5003-5027.

[11] - A. Strøm. *The homotopy category is a homotopy category.* Arch. Math. (Basel), 23:435-441, 1972.