

A CRITERION FOR A HUREWICZ COFIBRATION TO BE A QUILLEN COFIBRATION

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In this paper, we prove that h -cofibrations between q -cofibrant spaces are q -cofibrations and discuss a number of applications. In fact, since our non-equivariant proofs generalise readily to the equivariant context, we work throughout with the $\mathcal{C}$ -model structure on G -spaces. Here $\mathcal{C}$ is a set of subgroups of G which we take to be closed under conjugacy, and the $\mathcal{C}$ -model structure is that of [Sch18, Proposition B.7]. Recall that the $\mathcal{C}$ -model structure is a topological model structure cofibrantly generated by:

$$\begin{aligned} I &= \{G/H \times S^{n-1} \rightarrow G/H \times D^n \mid n \geq 0, H \in \mathcal{C}\}, \\ J &= \{G/H \times D^n \rightarrow G/H \times D^n \times I \mid n \geq 0, H \in \mathcal{C}\} \end{aligned}$$

Moreover, the $\mathcal{C}$ -fibrations (resp. $\mathcal{C}$ -equivalences) are the maps p such that p^H is a q -fibration (resp. q -equivalence) for every $H \in \mathcal{C}$. Note that our assumption that $\mathcal{C}$ is closed under conjugacy results in no loss of generality, and is convenient for some of the proofs below. For example, if H is a subgroup of G and we define $\mathcal{C} \cap H = \{K \cap H \mid K \in \mathcal{C}\}$, we then have:

Lemma 1: *If X is $\mathcal{C}$ -cofibrant as a G -space, then X is $(\mathcal{C} \cap H)$ -cofibrant as an H -space.*

Hurewicz cofibrations between q -cofibrant spaces are q -cofibrations

We now move onto the proof of our main theorem. The key point is the following generalisation of [MP12, Proposition 17.1.4], which admits the same proof:

Lemma 2: *Let $i : A \rightarrow B$ be an h -cofibration and $p : X \rightarrow Y$ be a map which has the RLP wrt $B \rightarrow B \times I$. Suppose that we are given a commutative square:*

$$\begin{array}{ccc} A & \xrightarrow{v} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{w} & Y \end{array}$$

and a map $\sigma : B \rightarrow X$ such that $p\sigma i = pv$, $\sigma i \simeq v$ over Y and $p\sigma \simeq w$ rel A . Then σ is homotopic to some τ with $\tau i = v$ and $p\tau = w$.

Proof. Since i is an h -cofibration, σ is homotopic to some σ' with $\sigma' i = v$ in such a way that the composite homotopy $p\sigma' \simeq w$ is still relative to A . So we have reduced to the case where $\sigma i = v$. Let (H, λ) represent (B, A) as an NDR -pair. Let $K : B \times I \rightarrow Y$ be a homotopy between $p\sigma$ and w , relative to A and normalised so that $K(b, t) = K(b, 1)$ for $t \geq \lambda(b)$. Since p has the RLP wrt $B \rightarrow B \times I$, we can lift this homotopy to $\tilde{K} : B \times I \rightarrow X$ starting at σ . If we define $\tilde{\sigma}(b) = \tilde{K}(b, \lambda(b))$ we solve the lifting problem as desired. $\square$

Lemma 3: *An h -cofibration between $\mathcal{C}$ -cofibrant spaces is a $\mathcal{C}$ -cofibration.*

Proof. Let $i : A \rightarrow B$ be an h -cofibration between $\mathcal{C}$ -cofibrant spaces. We can factor i as a $\mathcal{C}$ -cofibration followed by a $\mathcal{C}$ -acyclic $\mathcal{C}$ -fibration between $\mathcal{C}$ -cofibrant objects, so by the retract argument it suffices to show that i has the LLP wrt $\mathcal{C}$ -acyclic $\mathcal{C}$ -fibrations between $\mathcal{C}$ -cofibrant objects.

If $p : X \rightarrow Y$ is a $\mathcal{C}$ -acyclic $\mathcal{C}$ -fibration between $\mathcal{C}$ -cofibrant objects, then there exists $s : Y \rightarrow X$ such that $ps = 1$ and $sp \simeq 1$ over Y . Since B is $\mathcal{C}$ -cofibrant p has the RLP wrt $B \rightarrow B \times I$. Therefore, we can solve a lifting problem as in Lemma 2, with the initial up to homotopy lift σ as sw . $\square$

Example 4: Suppose that $H \in \mathcal{C}$. If U is an equivariant open subspace of $G/H \times D^n$, then $i : U \cap (G/H \times \partial D^n) \rightarrow U \cap (G/H \times D^n)$ is the inclusion of the boundary of a smooth G -manifold and so is an h -cofibration between $\mathcal{C}$ -cofibrant spaces by the equivariant collar neighbourhood theorem, [Kan07, Theorem 3.5]. Therefore, Lemma 3 tells us that i is a $\mathcal{C}$ -cofibration.

Lemma 5: If $i : A \rightarrow B$ is a $\mathcal{C}$ -cofibration and $U \subset B$ is an equivariant open subspace, then $i : A \cap U \rightarrow U$ is a $\mathcal{C}$ -cofibration.

Proof. Firstly, i is a retract of a relative $\mathcal{C}$ -cell complex, $j : A \rightarrow C$, as shown below:

$$\begin{array}{ccccc} A & \longrightarrow & A & \longrightarrow & A \\ \downarrow i & & \downarrow j & & \downarrow i \\ B & \xrightarrow{i} & C & \xrightarrow{r} & B \end{array}$$

If we let $V = r^{-1}(U)$, then $i : A \cap U \rightarrow U$ is a retract of $j : A \cap V \rightarrow V$. So we have reduced to the case where i is a relative $\mathcal{C}$ -cell complex, which we now assume.

We can express i as a transfinite composite of maps $A_\lambda \rightarrow A_{\lambda+1}$, each of which is a pushout of a map of the form $G/H \times S^{n-1} \rightarrow G/H \times D^n$, with $H \in \mathcal{C}$. By [Ron23, Lemma 6.2.3] and Lemma 21, the map $i : A \cap U \rightarrow U$ can be expressed as a transfinite composite of the maps $A_\lambda \cap U \rightarrow A_{\lambda+1} \cap U$, each of which is a pushout of a map of the form $(G/H \times S^{n-1}) \cap \alpha^{-1}(U) \rightarrow (G/H \times D^n) \cap \alpha^{-1}(U)$, where α is the inclusion of the cell into B . By Example 4, $A_\lambda \cap U \rightarrow A_{\lambda+1} \cap U$ is therefore a $\mathcal{C}$ -cofibration, and, hence, so is the transfinite composite $i : A \cap U \rightarrow U$. $\square$

Corollary 6: An equivariant open subspace of a $\mathcal{C}$ -cofibrant G -space is $\mathcal{C}$ -cofibrant.

Combining Lemma 3 and Corollary 6, we can rephrase Lemma 3 as follows:

Theorem 7: If $i : A \rightarrow B$ is an h -cofibration and B is $\mathcal{C}$ -cofibrant, then A is $\mathcal{C}$ -cofibrant and i is a $\mathcal{C}$ -cofibration.

Proof. If (H, λ) represents (B, A) as a G -NDR-pair, then A is a retract of $\lambda^{-1}([0, 1])$ which is $\mathcal{C}$ -cofibrant by Corollary 6. $\square$

Example 8: Let $G = \Sigma_n$ and let $\mathcal{C}$ be the set of subgroups of G , each isomorphic to a product of symmetric groups $\Sigma_{n_1} \times \dots \times \Sigma_{n_l}$, corresponding to partitions of $\{1, \dots, n\}$ into sets of size $n_1, \dots, n_l$. Let $i : S^{k-1} \rightarrow D^k$ be the boundary inclusion. If we consider the iterated pushout product map $j : S^{k-1} \times (D^k)^{n-1} \cup \dots \cup (D^k)^{n-1} \times S^{k-1} := Q_n(i) \rightarrow (D^k)^n$, then j is an h -cofibration by [May72, Lemma A.4], and the domain and codomain are $\mathcal{C}$ -cofibrant since they are G -manifolds, hence G -CW complexes, [Ill83, Corollary 7.2], with the correct isotropy groups. Therefore, Lemma ?? tells us that j is a $\mathcal{C}$ -cofibration.

We then have the following application to the theory of symmetrizable cofibrations, which is a slight generalisation of [Sch18, Proposition 2.1.12]:

Lemma 9: Let $j : A \rightarrow B$ be a symmetrizable (acyclic) cofibration in a tensored and cotensored closed symmetric monoidal topological model category. Then for every $k \geq 0$, the pushout product map:

$$j \square i : A \times D^k \cup_{A \times S^{k-1}} B \times S^{k-1} \rightarrow B \times D^k$$

is a symmetrizable (acyclic) cofibration.

Proof. For any $n \geq 1$, we want to show that $k := (j \square i)^{\square n} / \Sigma_n$ is an (acyclic) cofibration. Equivalently, we want to show that k has the Σ_n -equivariant LLP with respect to (acyclic) fibrations $p : X \rightarrow Y$, where Σ_n acts trivially on X and Y . Using the tensor-cotensor adjunction, it suffices to show that $i^{\square n}$ has the equivariant LLP wrt $p^{\square j^{\square n}}$. In Example 8, we saw that $i^{\square n}$ is a $\mathcal{C}$ -cofibration, with respect to the specified set of subgroups $\mathcal{C}$. So it suffices to show that $p^{\square j^{\square n}}$ is a $\mathcal{C}$ -acyclic $\mathcal{C}$ -fibration. If $H \in \mathcal{C}$, this amounts to showing $p^{\square j^{\square n} / H}$ is a q -acyclic q -fibration. This follows from the fact that the model structure is topological and monoidal, j is a symmetrizable cofibration and either p is acyclic or j is a symmetrizable acyclic cofibration. $\square$

Local gluing results for fibrations and cofibrations

In this subsection, we explain how Theorem 7 allows us to prove gluing results for $\mathcal{C}$ -fibrations and $\mathcal{C}$ -cofibrations analogous to Dold's gluing results for h -fibrations and h -cofibrations, [Dol63, Theorem 4.8] and [Dol68, Satz 2]. First recall that:

Definition 10: A G -numerable open cover $\{U_j\}_{j \in J}$ of a G -space, B , is a locally finite open cover of B such that for every $j \in J$ there exists an equivariant map $\lambda_j : B \rightarrow I$ with $U_j = \lambda_j^{-1}((0, 1])$.

The next lemma is central to the proofs of all four gluing theorems:

Lemma 11: Consider a lifting problem of G -spaces as below:

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

Suppose that there exists a G -numerable open cover $\{U_i\}_{i \in I}$ of B such that if $i \in I$ and $J \subset I$, then there exists a solution to the induced lifting problem:

$$\begin{array}{ccc} i^{-1}(U_i \cap (\cup_{j \in J} U_j)) & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ U_i \cap (\cup_{j \in J} U_j) & \xrightarrow{g} & Y \end{array}$$

and given two solutions ψ_1, ψ_2 of the induced lifting problem there exists a G -homotopy from ψ_1 to ψ_2 which is relative to both p and i .

Then there is a solution to the original lifting problem 1.

Proof. Let:

$$S = \{(A, \psi) \mid A \subset I, \psi : \cup_{\alpha \in A} U_\alpha \rightarrow X, p\psi = g, \psi i = f\}.$$

We define a partial order on S by:

$$(A, \psi_1) \leq (B, \psi_2)$$

iff $A \subset B$ and $\psi_1 = \psi_2$ on $\cup_{\alpha \in A} U_\alpha \setminus \cup_{\beta \in B \setminus A} U_\beta$.

We may as well assume that B is non-empty, in which case taking $J = \{i\}$ in the condition of the lemma shows that S is non-empty. Since $\{U_i\}$ is a locally finite cover, every chain in S has an upper bound. Hence, by Zorn's lemma, S contains a maximal element, say (A, ψ_1) . If $A \neq I$, there exists some $\beta \in I \setminus A$. By the condition in the lemma, there exists some $(\{\beta\}, \psi_2) \in S$. By the uniqueness up to relative homotopy condition, there exists a G -homotopy $K : (U_\beta \cap (\cup_{\alpha \in A} U_\alpha)) \times I \rightarrow X$ between ψ_1 and ψ_2 , relative to p and i . Pick G -maps $\lambda_1, \lambda_2 : U_\beta \cup (\cup_{\alpha \in A} U_\alpha) \rightarrow I$ such that $\cup_{\alpha \in A} U_\alpha = \lambda_1^{-1}((0, 1])$, $U_\beta = \lambda_2^{-1}((0, 1])$ and $\lambda_1 + \lambda_2 = 1$. Define:

$$\psi(u) = \begin{cases} \psi_1(u) & \text{if } \lambda_1 > \frac{3}{4} \\ \psi_2(u) & \text{if } \lambda_1 < \frac{1}{4} \\ K(u, \frac{3}{2} - 2\lambda_1) & \text{otherwise} \end{cases}$$

Then $(A \cup \{\beta\}, \psi) \in S$ with $(A, \psi_1) \leq (A \cup \{\beta\}, \psi)$, a contradiction. So $A = I$ and ψ_1 is a solution to the original lifting problem. $\square$

As an immediate consequence, we have the following gluing result for $\mathcal{C}$ -cofibrations:

Theorem 12: If $i : A \rightarrow B$ is a map of G -spaces and there is a G -numerable open cover $\mathcal{U}$ of B such that for every $U \in \mathcal{U}$, $i|_{A \cap U} : A \cap U \rightarrow U$ is a $\mathcal{C}$ -cofibration, then i is a $\mathcal{C}$ -cofibration.

Proof. We consider a lifting problem for i with respect to a $\mathcal{C}$ -acyclic $\mathcal{C}$ -fibration, $p : X \rightarrow Y$. The G -numerable open cover $\mathcal{U}$ ensures the existence of local lifts as in Lemma 11. Uniqueness follows from Lemma 5 and the fact that the $\mathcal{C}$ -model structure is topological, [Sch18, Proposition B.5], so solutions to such lifting problems are unique up to relative homotopy. Therefore, the result follows from Lemma 11. $\square$

The corresponding gluing theorem for h -cofibrations is [Dol68, Satz 2], whose proof uses the fact that the restriction of an h -cofibration to a numerable open subspace is an h -cofibration, [Dol68, Satz 1], in place of Lemma 5:

Theorem 13: *Suppose that $\{U_j\}_{j \in J}$ is a G -numerable open cover of B . If $f : A \rightarrow B$ is a map such that $f : f^{-1}(U_j) \rightarrow U_j$ is an hG -cofibration for all $j \in J$, then $f : A \rightarrow B$ is an hG -cofibration.*

As for h -fibrations, [Dol63, Theorem 4.8], with a little more work we can use Lemma 11 to prove a gluing result for $\mathcal{C}$ -fibrations:

Theorem 14: *If $p : X \rightarrow Y$ is a map and there exists a G -numerable open cover $\{U_i\}_{i \in I}$ of Y such that $p : p^{-1}(U_i) \rightarrow U_i$ is a $\mathcal{C}$ -fibration for all $i \in I$, then p is a $\mathcal{C}$ -fibration.*

Proof. Consider a lifting problem:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ A \times I & \xrightarrow{H} & Y \end{array}$$

where A is $\mathcal{C}$ -cofibrant and $i : A \rightarrow A \times I$ is the inclusion of $A \times \{0\}$.

Choose functions, $\lambda_i : Y \rightarrow I$, for every $i \in I$, such that $U_i = \lambda_i^{-1}((0, 1])$ and $\sum_{i \in I} \lambda_i = 1$. Suppose that $S = (i_0, \dots, i_{n-1})$ is a finite sequence of elements of I , so, in particular, the members of the sequence need not be distinct. Define a function $\mu_S : A \rightarrow I$ by:

$$(2) \quad \mu_S(a) = \max \left\{ \min_{j \in \{0, \dots, n-1\}} \left(\inf_{t \in [\frac{j}{n}, \frac{j+1}{n}]} \lambda_{i_j}(H(a, t)) \right) - n \sum_{T \mid |T| < n} \mu_T, 0 \right\}$$

Then $\{V_S := \mu_S^{-1}((0, 1])\}$ is a G -numerable open cover of A , indexed over finite sequences of elements of I . Therefore, $\{V_S \times I\}$ is a G -numerable open cover of $A \times I$. Moreover, by inspection of equation (2), we see that $H(V_S \times [\frac{j}{n}, \frac{j+1}{n}]) \subset U_{i_j}$ for every j . It follows that the existence condition of Lemma 11 is satisfied, since V_S is $\mathcal{C}$ -cofibrant by Corollary 6 and so we can construct a lift inductively on these intervals. Similarly, we can construct relative homotopies between any two lifts inductively, and so the uniqueness condition of Lemma 11 is also satisfied, which gives the result. $\square$

Other Examples

The next result is the equivariant analogue of [DE72, Corollary III.2].

Lemma 15: *Suppose that $\mathcal{C}$ is closed under conjugacy and intersections. If X is $\mathcal{C}$ -cofibrant, then $\Delta : X \rightarrow X \times X$ is a $\mathcal{C}$ -cofibration.*

Proof. The condition that $\mathcal{C}$ is closed under conjugacy and intersections ensures that $X \times X$ is also $\mathcal{C}$ -cofibrant. Since any $\mathcal{C}$ -cofibrant G -space is a retract of a $\mathcal{C}$ -cell complex, by Lemma 3 it suffices to show that if X is a G -cell complex then $\Delta : X \rightarrow X \times X$ is an h -cofibration. To this end, assume that $\Delta : Y \rightarrow Y \times Y$ is an h -cofibration and X is obtained from Y by attaching a single cell along $\alpha : G/H \times S^{n-1} \rightarrow Y$. We will show that if (H, λ) represents $(Y \times Y, Y)$ an NDR-pair, then there is an NDR pair (K, μ) for $(X \times X, X)$ such that $K|_{Y \times Y \times I} = H$ and $\mu|_{Y \times Y} = \lambda$. Firstly, the composite $Y \rightarrow Y \times Y \rightarrow Y \times X \cup_{Y \times Y} X \times Y$ is an h -cofibration, represented by an NDR pair that extends (H, λ) , where note $Y \times Y \rightarrow Y \times X \cup_{Y \times Y} X \times Y$ is an h -cofibration by [Str72, Lemma 5]. We then consider the map between pushouts:

$$\begin{array}{ccccc}
G/H \times D^n & \xleftarrow{\quad} & G/H \times S^{n-1} & \xrightarrow{\quad} & Y \\
\downarrow \Delta & & \downarrow \Delta & & \downarrow \alpha \\
G/H \times G/H \times D^n \times D^n & \xleftarrow{\quad} & G/H \times G/H \times \partial D^{2n} & \longrightarrow & Y \times X \cup_{Y \times Y} X \times Y
\end{array}$$

We've seen that α is an h -cofibration. The map between pushouts is $\Delta : X \rightarrow X \times X$ and to show this is an h -cofibration it suffices to show the pushout-product map, ϕ , below is an h -cofibration:

$$\phi : G/H \times D^n \cup_{G/H \times S^{n-1}} G/H \times G/H \times \partial D^{2n} \rightarrow G/H \times G/H \times D^n \times D^n$$

Let $U \cong G/H \times \partial D^n \times [0, 1)$ be a numerable boundary collar in $G/H \times D^n$. Then consider the restriction of ϕ to $U \times U$, which can be identified with:

$$G/H \times \partial D^n \times [0, 1) \cup G/H \times G/H \times \partial D^n \times \partial D^n \times \partial([0, 1) \times [0, 1)) \rightarrow G/H \times G/H \times \partial D^n \times \partial D^n \times [0, 1) \times [0, 1)$$

Let $L \subset [0, 1) \times [0, 1)$ be the union of the diagonal and the boundaries, so L consists of three straight lines. The map above is the composite of the pushout product of $\Delta : G/H \times \partial D^n \rightarrow G/H \times G/H \times \partial D^n \times \partial D^n$ and $\partial([0, 1) \times [0, 1)) \rightarrow L$ with the map:

$$G/H \times G/H \times \partial D^n \times \partial D^n \times L \rightarrow G/H \times G/H \times \partial D^n \times \partial D^n \times [0, 1) \times [0, 1)$$

and is therefore an h -cofibration as desired. If V is the complement in $G/H \times D^n$ of $G/H \times \partial D^n \times [0, 1/2] \subset U$, then V is also numerable and the restriction of ϕ to each of $U \times U$, $V \times G/H \times D^n$ and $G/H \times D^n \times V$ is an h -cofibration. Therefore, ϕ is an h -cofibration by [Dol68, Satz 2], and, therefore, $\Delta : X \rightarrow X \times X$ is an h -cofibration, since it is the composite of a pushout of α with a pushout of ϕ . Etc... $\square$

As in [DE72], we make the following definition:

Definition 16: We call a G -space, X , G -equivariantly locally equiconnected (or G -LEC) if $\Delta : X \rightarrow X \times X$ is an hG -cofibration.

The next lemma is the equivariant analogue of [DE72, Theorem II.3]:

Lemma 17: Let X be G -LEC and suppose that $x \in X$ has isotropy group H . Then there exists an H -equivariant open neighbourhood U of x such that $\pi_1, \pi_2 : U \times U \rightarrow X$ are H -homotopic. Here, π_i denotes the projection onto the i th factor followed by the inclusion into X .

Proof. Let (H, λ) represent the inclusion of the diagonal $(X \times X, \Delta)$ as a G -NDR pair. Identify X with $\{x\} \times X$ and let $U = \lambda^{-1}([0, 1)) \cap (\{x\} \times X)$. Consider $H : \{x\} \times U \times I \rightarrow X \times X$. Then $\pi_2 H$ defines an H -homotopy from the inclusion $i : U \rightarrow X$ to some map f . Moreover, $\pi_1 H$ defines an H -homotopy from f to the constant map to x . If we now consider $\pi_1, \pi_2 : U \times U \rightarrow X$, we see that π_1 is the composite $U \times U \rightarrow X \times U \xrightarrow{\pi_1} X$, which is H -homotopic to the constant map to x . Similarly, π_2 is H -homotopic to the constant map to x , so π_1 and π_2 are H -homotopic. $\square$

Lemma 18: Let X be $\mathcal{C}$ -cofibrant and suppose that $x \in X$ has isotropy group H . Then there exists an H -equivariant open neighbourhood U of x such that $\pi_1, \pi_2 : U \times U \rightarrow X$ are H -homotopic via an H -homotopy which is constant on Δ .

Proof. By Lemma 15 and Lemma 17, we know that π_1 and π_2 are H -homotopic, say via an H -homotopy $L : U \times U \times I \rightarrow X$. Define $\Gamma : \Delta \times I \rightarrow X$ to be the restriction of L to $\Delta \times I$. Let Γ^{-1} denote the inverse homotopy from π_2 to π_1 . Define K to be composite (via concatenation) homotopy of L and $\Gamma^{-1}\pi_2$:

$$U \times U \times I \xrightarrow{\pi_2} U \times I \cong \Delta \times I \xrightarrow{\Gamma^{-1}} X$$

Then the restriction of K to $\Delta \times I$ is the composite of Γ and Γ^{-1} , and K is an H -homotopy from π_1 to π_2 . The composite of Γ and Γ^{-1} is itself H -homotopic, say via $\alpha : \Delta \times I \times I \rightarrow X$, to the constant homotopy on $\pi_1|_\Delta$, and α can be made relative to $\Delta \times \{0, 1\} \times I$. We can then define a lifting problem:

$$\begin{array}{ccc}
U \times U \times I \times \{0\} \cup \Delta \times I \times I \cup U \times U \times \{0, 1\} \times I & \xrightarrow{K \cup \alpha \cup \text{const}(\{\pi_1, \pi_2\})} & X \\
\downarrow & & \downarrow \\
U \times U \times I \times I & \xrightarrow{\quad\quad\quad} & *
\end{array}$$

A lift $\tilde{K} : U \times U \times I \times I \rightarrow X$ exists since U is $(\mathcal{C} \cap H)$ -cofibrant, by Corollary 6 and Lemma 1, and, therefore, H -LEC by Lemma 15. This implies the left hand vertical map is an hH -acyclic hH -cofibration. Restricting $\tilde{K}$ to $U \times U \times I \times \{1\}$ gives the desired H -homotopy. $\square$

We will use the following consequence of [Las82, Lemma 2.8], which was noted in [BJ15, Lemma 6.1]:

Lemma 19: *If $p : E \rightarrow B$ is an hH -fibration, then $1 \times_H p : G \times_H E \rightarrow G \times_H B$ is an hG -fibration.*

Proof. Consider a lifting problem of G -spaces as in the right hand square below:

$$\begin{array}{ccccc}
G \times_H V_0 & \xrightarrow{\cong} & X & \xrightarrow{f} & G \times_H E \\
\downarrow & & \downarrow & & \downarrow \\
G \times_H (V_0 \times I) & \xrightarrow{\cong} & X \times I & \xrightarrow{\alpha} & G \times_H B
\end{array}$$

Then $X \times I \cong G \times_H \alpha^{-1}(B)$ and so [Las82, Lemma 2.8] tells us that there is an H -homeomorphism $\psi : V_0 \times I \rightarrow V$, where $V = \alpha^{-1}(B)$, $V_0 = V \cap (X \times \{0\})$ and $\psi|_{V_0 \times \{0\}}$ is the inclusion of V_0 into V . Note that $G \times_H V_0 \cong X$. The H -homeomorphism, ψ , induces a G -homeomorphism $G \times_H (V_0 \times I) \rightarrow X \times I$, as in the diagram above, which restricts at $0 \in I$ to a homeomorphism onto $X \times \{0\}$. So to solve the lifting problem of the right hand square, it suffices to solve the adjoint lifting problem of H -spaces:

$$\begin{array}{ccc}
V_0 & \xrightarrow{f} & G \times_H E \\
\downarrow & & \downarrow \\
V_0 \times I & \xrightarrow{\alpha\psi} & G \times_H B
\end{array}$$

Since $V = \alpha^{-1}(B)$, $\alpha\psi : V_0 \times I \rightarrow G \times_H B$ factors through B , and $f : V_0 \rightarrow G \times_H E$ factors through E . Since p is an hH -fibration, it follows that the lifting problem can be solved. $\square$

A version of the following theorem was first proved in an equivariant context by Gevorgyan and Jimenez in [GJ19, Theorem 6], using the qG -model structure and under the assumption that both E and B are G -CW complexes. The proof below corrects a minor gap in that proof, since it is not necessarily true that a G -CW complex, X , can be covered by G -equivariant open subspaces which are non-equivariantly contractible in X .

Theorem 20: *Suppose that $\mathcal{C}$ is closed under conjugacy and intersections. If $p : E \rightarrow B$ is a $\mathcal{C}$ -fibration between $\mathcal{C}$ -cofibrant spaces, then p is an hG -fibration.*

Proof. Let (H, λ) represent $(B \times B, \Delta_B)$ as a G -NDR pair. As in Lemma 18, if $b \in B$ has isotropy group H there is an H -equivariant open neighbourhood U of b and an H -homotopy $K : U \times U \times I \rightarrow B$ between π_1 and π_2 such that $K(u, v, t) = K(u, v, 1)$ for all $t \geq \lambda(u, v)$. Note that $H \in \mathcal{C}$ since there exists an inclusion of B into a $\mathcal{C}$ -cell complex. Since $p : E \rightarrow B$ is a $(\mathcal{C} \cap H)$ -fibration, $\mathcal{C}$ is closed under intersections and both U and $p^{-1}(U)$ are $(\mathcal{C} \cap H)$ -cofibrant, we can solve the lifting problem:

$$\begin{array}{ccc}
U \times p^{-1}(U) & \xrightarrow{\pi_2} & E \\
\downarrow & & \downarrow p \\
U \times p^{-1}(U) \times I & \xrightarrow{K \circ (1 \times p \times 1)} & B
\end{array}$$

to define a slicing function $\tilde{K} : U \times p^{-1}(U) \times I \rightarrow E$. We now show that $p : p^{-1}(U) \rightarrow U$ is an hH -fibration. Indeed, given a lifting problem of H -spaces:

$$\begin{array}{ccc}
X & \xrightarrow{f} & p^{-1}(U) \\
\downarrow & & \downarrow p \\
X \times I & \xrightarrow{L} & U
\end{array}$$

a lift can be defined via $\tilde{L}(x, t) = \tilde{K}(L(x, t), f(x), \lambda(L(x, t), pf(x)))$.

Since B is completely regular, b also has a G -equivariant open neighbourhood G -homeomorphic to $G \times_H S$, where S is an H -equivariant subspace of B containing b , by [Bre72, Theorem 5.4]. Replacing S by $S \cap U$ if necessary, we may as well assume $S \subset U$. In this case, it follows from the above that $p : p^{-1}(S) \rightarrow S$ is an hH -fibration and so $p : G \times_H p^{-1}(S) \rightarrow G \times_H S$ is an hG -fibration by Lemma 19. It follows that for every $b \in B$ there is a G -equivariant neighbourhood, V , of b for which $p : p^{-1}(V) \rightarrow V$ is an hG -fibration. Since B is paracompact, it follows that p is an hG -fibration. $\square$

Appendix of point set topology

Lemma 21: *Let I be a filtered category and let $D : I \rightarrow \mathcal{U}$ be a functor which takes maps in I to closed inclusions. Let $A := \text{colim} D$ and $f : X \rightarrow A$ be a map. Define $Y(i)$ to be the pullback of $D(i) \rightarrow A$ along f . Then the canonical map $\text{colim} Y \rightarrow X$ is a homeomorphism.*

Proof. We can express filtered colimits of closed inclusions, such as A , as quotients of the disjoint union of the colimiting subspaces, $\sqcup_{i \in I} D(i) \rightarrow A$, [Str09, Lemma 3.3]. We have a pullback:

$$\begin{array}{ccc}
\sqcup_{i \in I} Y_i & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\sqcup_{i \in I} D_i & \longrightarrow & A
\end{array}$$

Since the pullback of a quotient map is a quotient map, [Str09, Propostion 2.36], we obtain the result. $\square$

Lemma 22: *The following properties of a space X are closed under retracts: being i) Hausdorff, ii) paracompact, iii) completely regular.*

Proof. Suppose that we have maps $i : A \rightarrow X, r : X \rightarrow A$ with $ri = 1$. If X is Hausdorff, let a and b be distinct points in A . Then there exists disjoint open sets U and V in X separating $i(a)$ and $i(b)$. So $i^{-1}(U)$ and $i^{-1}(V)$ are disjoint and separate a and b . So A is Hausdorff.

If X is paracompact and $\{U_i\}_{i \in I}$ is an open cover of A , then $\{r^{-1}(U_i)\}$ is an open cover of X which admits a locally finite open refinement $\{V_j\}_{j \in J}$. Then, for each j $i^{-1}(V_j)$ is contained within $i^{-1}r^{-1}(U_i) = U_i$ for some i , so $\{i^{-1}(V_j)\}$ is an open refinement of $\{U_i\}$. Moreover, for every $a \in A$ there is an open neighbourhood, W , of $i(a)$ which intersects only finitely many of the V_j . Then $i^{-1}(W)$ is an open neighbourhood of a , intersecting only finitely many of the $i^{-1}(V_j)$. So A is paracompact.

Next suppose that X is completely regular. Let $a \in A$ and C be a closed subspace of A not containing a . Then $r^{-1}(C)$ is a closed subspace of X not containing $i(a)$. So there exists a continuous function $\lambda : X \rightarrow [0, 1]$ such that $\lambda i(a) = 0$ and $\lambda(r^{-1}(C)) = 1 \implies \lambda i(C) = 1$. So A is completely regular. $\square$

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